

# MAPPING SPACES AND $R$ -COMPLETION

DAVID BLANC AND DEBASIS SEN

**ABSTRACT.** We study the questions of how to recognize when a simplicial set  $X$  is of the form  $X = \mathrm{map}_*(\mathbf{Y}, \mathbf{A})$ , for a given space  $\mathbf{A}$ , and how to recover  $\mathbf{Y}$  from  $X$ , if so. A full answer is provided when  $\mathbf{A} = \mathbf{K}(R, n)$ , for  $R = \mathbb{F}_p$  or  $\mathbb{Q}$ , in terms of a *mapping algebra* structure on  $X$  (defined in terms of product-preserving simplicial functors out of a certain simplicially-enriched sketch  $\Theta_{\mathbf{A}}$ ). In addition, when  $\mathbf{A}$  is a suitable  $\Omega^\infty$ -space – such as  $\mathbf{K}(R, n)$  for any commutative ring  $R$  – we can *recover*  $\mathbf{Y}$  from  $\mathrm{map}_*(\mathbf{Y}, \mathbf{A})$ , given such a mapping algebra structure. Along the way, we observe that our methods provide a new way of looking at the classical Bousfield-Kan  $R$ -completion.

## INTRODUCTION

Given pointed topological spaces  $\mathbf{A}$  and  $\mathbf{Y}$ , one can form the simplicial mapping space  $\mathrm{map}_*(\mathbf{Y}, \mathbf{A})$ , which models the topological space  $\mathrm{Hom}_{\mathcal{T}_*}(\mathbf{Y}, \mathbf{A})$  of all pointed continuous maps (with the compact-open topology). Such mapping spaces play a central role in modern homotopy theory, so it is natural to ask when a simplicial set  $X$  is of the form  $\mathrm{map}_*(\mathbf{Y}, \mathbf{A})$ , up to weak equivalence. We assume that one of the two spaces  $\mathbf{A}$  and  $\mathbf{Y}$  is given, but not both.

This question has been extensively studied in the case where  $\mathbf{Y} = \mathbf{S}^n$  is a sphere, so  $X$  is an  $n$ -fold loop space (see, e.g., [Su, Sta, M2, Ba]). A general answer for any pointed  $\mathbf{Y}$  was provided in [BBD]. In this paper we consider the dual question:

Given a space  $\mathbf{A}$  and a simplicial set  $X$ ,

- (a) when is  $X$  of the form  $\mathrm{map}_*(\mathbf{Y}, \mathbf{A})$  for some space  $\mathbf{Y}$ ?
- (b) If  $X$  satisfies the conditions prescribed in the answer to (a), how can we recover  $\mathbf{Y}$  from it?

Note that the best we can hope for is to recover  $\mathbf{Y}$  up to  $\mathbf{A}$ -equivalence (where a map  $f : \mathbf{Y} \rightarrow \mathbf{Y}'$  in a simplicial category  $\mathcal{C}$  is called an  $\mathbf{A}$ -equivalence if it induces a weak equivalence  $\mathrm{map}_{\mathcal{C}}(\mathbf{Y}', \mathbf{A}) \sim \mathrm{map}_{\mathcal{C}}(\mathbf{Y}, \mathbf{A})$ ).

Unfortunately, the methods of [BBD] do not carry over in full to the dual problem: we can answer both questions only when  $\mathbf{A} = \mathbf{K}(R, n)$  for  $R = \mathbb{Q}$  or  $\mathbb{F}_p$ . However, if we know that  $X$  is a mapping space, and only wish to recover  $\mathbf{Y}$ , we can do a little better: in this case we can allow  $\mathbf{A}$  to be a  $\Omega^\infty$ -space model for a suitable ring spectrum (so in particular,  $\mathbf{A}$  can be  $\mathbf{K}(R, n)$  for any commutative ring  $R$ ).

**0.1. Remark.** Of course,  $\mathrm{map}_*(\mathbf{Y}, \mathbf{K}(R, n))$  is itself an  $R$ -GEM – i.e., it is homotopy equivalent to a product of  $R$ -module Eilenberg-Mac Lane spaces – so it appears to carry very little homotopy-invariant information. In particular, any finite type  $\mathbb{Z}$ -GEM  $X \in \mathcal{S}_*$  (that is,  $X \simeq \prod_{i=1}^n \mathbf{K}(M_i, i)$  for finitely-generated abelian groups

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*Date:* April 23, 2013.

*2010 Mathematics Subject Classification.* Primary: 55P48; secondary: 55P20, 55P60, 55U35.

*Key words and phrases.* Mapping space, mapping algebra,  $p$ -completion, rationalization, cosimplicial resolution.

$M_0, \dots, M_n$ ) is homotopy equivalent to  $\text{map}_*(\mathbf{Y}, \mathbf{K}(\mathbb{Z}, n))$  for some  $\mathbf{Y} \in \mathcal{T}_*$  – e.g., for  $\mathbf{Y} \simeq \bigvee_{i=0}^n L(M_i, i)$ , where  $L(M_i, i)$  is a co-Moore space (cf. [Ka]). Thus the answer to (a) is always positive, for such an  $X$ .

There are two difficulties with this point of view:

- (i) This will not necessarily work for other rings  $R$ , or if the abelian groups  $M_i$  are not finitely generated (see [GG] and §5.5 below).
- (ii) Moreover, there are usually many other possible choices of  $\mathbf{Y}$ , and the homotopy type of  $X$  alone does not enable us to distinguish between them.

With reference to the second point, we must therefore impose sufficient additional structure on  $X = \text{map}_*(\mathbf{Y}, \mathbf{A})$  to allow us to recover  $\mathbf{Y}$ , uniquely up to  $\mathbf{A}$ -equivalence. This structure is defined as follows:

**0.2. Mapping algebras.** In the original example of loop space recognition mentioned above, the additional data on  $\Omega\mathbf{Y} = \text{map}_*(\mathbf{S}^1, \mathbf{Y})$  consisted of the group structure, induced by the cogroup structure map  $\nabla : \mathbf{S}^1 \rightarrow \mathbf{S}^1 \vee \mathbf{S}^1$ , its iterates, and various (higher) homotopies between them. These are all encoded in the actions, by composition, of pointed mapping spaces between wedges of circles on  $\Omega\mathbf{Y}$  (and on  $\Omega\mathbf{Y} \times \Omega\mathbf{Y} = \text{map}_*(\mathbf{S}^1 \vee \mathbf{S}^1, \mathbf{Y})$ , and so on). A codification of this additional structure (for a fixed space  $\mathbf{Y}$ ) was provided in [BB1, BBD].

This suggests that in the dual case one should look the action of the mapping spaces  $\text{map}_*(\prod_{i=1}^n \mathbf{A}, \mathbf{A})$  on  $\text{map}_*(\mathbf{Y}, \prod_{i=1}^n \mathbf{A}) = \prod_{i=1}^n \text{map}_*(\mathbf{Y}, \mathbf{A})$ . Moreover, since  $\Omega \text{map}_*(\mathbf{Y}, \mathbf{A}) = \text{map}_*(\mathbf{Y}, \Omega\mathbf{A})$ , we should consider also maps products of various loop spaces of  $\mathbf{A}$ .

Although some of our results are valid for any  $H$ -group  $\mathbf{A} \in \mathcal{T}_*$ , their full force requires the ability to deloop  $\mathbf{A}$  arbitrarily, so we start with an  $\Omega$ -spectrum  $\mathcal{A} = (\underline{\mathbf{A}}_n)_{n \in \mathbb{Z}}$ , having  $\mathbf{A} = \underline{\mathbf{A}}_0$ , say, in mind for our original questions (a) and (b). We may then formalize the additional structure mentioned above as follows:

Let  $\Theta_{\mathcal{A}}$  be the sub-category of  $\mathcal{T}_*$  whose objects are generated by the spaces  $\underline{\mathbf{A}}_n$  ( $n \in \mathbb{Z}$ ) under products of cardinality  $< \lambda$ , for a suitably chosen cardinal  $\lambda$ . Thus the objects have the form  $\prod_{i \in I} \underline{\mathbf{A}}_{n_i}$ , for some  $(n_i)_{i \in I}$ . This will be called an *enriched sketch*, and it inherits a simplicial enrichment from  $\mathcal{T}_*$ .

For example, if  $\mathcal{A} = (\mathbf{K}(R, n))_{n=0}^{\infty}$  is the  $R$ -Eilenberg-Mac Lane spectrum and  $\lambda = \aleph_0$ , then  $\Theta_{\mathcal{A}}$  consists of all finite type  $R$ -GEMs.

A simplicial functor  $\mathfrak{X} : \Theta_{\mathcal{A}} \rightarrow \mathcal{S}_*$  which preserved all loops and products will be called an  $\Theta_{\mathcal{A}}$ -mapping algebra. The main example we have in mind is a *realizable*  $\Theta_{\mathcal{A}}$ -mapping algebra, of the form  $\mathbf{B} \mapsto \text{map}_*(\mathbf{Y}, \mathbf{B})$  for some fixed  $\mathbf{Y} \in \mathcal{T}_*$  (and all  $\mathbf{B} \in \Theta_{\mathcal{A}}$ , of course).

Note that if  $X = \text{map}_*(\mathbf{Y}, \underline{\mathbf{A}}_k)$ , the realizable  $\Theta_{\mathcal{A}}$ -mapping algebra  $\mathfrak{X}$  corresponding to  $\mathbf{Y}$  has  $\mathfrak{X}(\prod_{i \in I} \underline{\mathbf{A}}_{n_i}) = \prod_{i \in I} \Omega^{k-n_i} X$ , so we can think of  $\mathfrak{X}$  as additional structure on the simplicial set  $X$ , including choices of *deloopings* of  $X$ .

It turns out that this structure is precisely what is needed to recover  $\mathbf{Y}$  from  $X$ , under suitable assumptions. In fact, we show:

**Theorem A.** *Let  $\mathcal{A} = (\underline{\mathbf{A}}_n)_{n=0}^{\infty}$  be an  $\Omega$ -spectrum model of a connective ring spectrum, with  $\pi_0 \mathcal{A}$  commutative, and let  $\mathfrak{X}$  be a  $\mathcal{A}$ -Stover mapping algebra structure (cf. §3.21) on  $X = \text{map}_*(\mathbf{Y}, \underline{\mathbf{A}}_0)$  for a simply-connected space  $\mathbf{Y}$ . We then can construct functorially a cosimplicial space  $\mathbf{W}^\bullet$  with total space  $\text{Tot } \mathbf{W}^\bullet$  weakly equivalent to the  $\mathcal{A}$ -completion of  $\mathbf{Y}$  (so that  $\text{Tot } \mathbf{W}^\bullet$  realizes  $X$ ).*

See Theorem 3.27, Corollary 3.33 and Proposition 3.35 below. The  $\mathcal{A}$ -completion here is just the usual  $R$ -completion of  $\mathbf{Y}$  for  $R = \text{core } \pi_0 \mathcal{A}$  (cf. [BK2]).

**0.3.  $\Theta$ -algebras.** If  $\Theta_{\mathcal{A}}$  is an enriched sketch as above, applying  $\pi_0$  to each of its mapping spaces yields a category  $\Theta_{\mathcal{A}} := \pi_0 \Theta_{\mathcal{A}}$ , which is an ordinary “algebraic” *sketch* in the sense of Ehresmann (see §2.1). If  $\mathfrak{X} : \Theta_{\mathcal{A}} \rightarrow \mathcal{S}_*$  is a  $\Theta_{\mathcal{A}}$ -mapping algebra, composing with  $\pi_0$  yields an  $\Theta_{\mathcal{A}}$ -algebra  $\Lambda := \pi_0 \mathfrak{X}$  – that is, a product-preserving functor  $\Lambda : \Theta_{\mathcal{A}} \rightarrow \mathbf{Set}_*$ . We can think of such an  $\Theta_{\mathcal{A}}$ -algebra as an algebraic version of an  $\Theta_{\mathcal{A}}$ -mapping algebra. Note that the category of simplicial  $\Theta_{\mathcal{A}}$ -algebras has a model category structure as in [Q, II, §4], which allows us to define free simplicial resolutions of  $\Lambda$ .

For example, if  $\mathcal{A} = (\mathbf{K}(n, \mathbb{F}_p))_{n=0}^{\infty}$  (and  $\lambda = \aleph_0$  as above),  $\Theta_{\mathcal{A}}$  is the homotopy category of finite type  $\mathbb{F}_p$ -GEMs, and a  $\Theta_{\mathcal{A}}$ -algebra is just an algebra over the mod  $p$  Steenrod algebra, as in [Sc, §1.4].

Our main technical tool in this paper, which we hope will be of independent use, is the following:

**Theorem B.** *If  $\mathfrak{X}$  is an  $\Theta_{\mathcal{A}}$ -mapping algebra and  $\Lambda = \pi_0 \mathfrak{X}$  is the corresponding  $\Theta_{\mathcal{A}}$ -algebra, then any algebraic CW-simplicial resolution  $V_{\bullet} \rightarrow \Lambda$  (cf. §4.1) can be realized by a cosimplicial space  $\mathbf{W}^{\bullet}$ .*

See Theorem 4.6 below.

**0.4. Warning.** We do not claim that the space  $\mathbf{Y} := \text{Tot } \mathbf{W}^{\bullet}$  obtained by means of Theorem B in fact realizes the  $\Theta_{\mathcal{A}}$ -mapping algebra  $\mathfrak{X}$ . In particular, if  $\mathfrak{X}$  is an  $\Theta_{\mathcal{A}}$ -mapping algebra structure on a simplicial set  $X$ , it need not be true that  $X \simeq \text{map}_*(\mathbf{Y}, \mathbf{A})$ , even for  $\mathbf{A} = \mathbf{K}(R, n)$  (see §5.5 below).

However, it turns out that, for suitable fields  $R$ , the only obstruction to realizability is a purely algebraic condition on  $\Lambda = \pi_0 \mathfrak{X}$ . In particular, we show:

**Theorem C.** *When  $\mathcal{A} = (\mathbf{K}(R, n))_{n=0}^{\infty}$  for  $R = \mathbb{Q}$  or  $\mathbb{F}_p$ , any simply-connected finite type  $\Theta_{\mathcal{A}}$ -mapping algebra  $\mathfrak{X}$  is realizable by an  $R$ -complete space  $\mathbf{Y}$ , unique up to  $R$ -equivalence.*

See Theorem 5.13 below (which is somewhat more general).

Theorem C implies that for  $R = \mathbb{Q}$  or  $\mathbb{F}_p$ , a finite type  $R$ -GEM  $X$  which can be endowed with a simply-connected  $\Theta_{\mathcal{A}}$ -mapping algebra structure (also involving deloopings of  $X$ ) is realizable as a mapping space  $X \simeq \text{map}_*(\mathbf{Y}, \mathbf{K}(R, n))$  for some  $R$ -complete  $\mathbf{Y}$ . Moreover,  $\mathbf{Y}$  is uniquely determined up to  $R$ -equivalence by the choice of  $\Theta_{\mathcal{A}}$ -mapping algebra structure.

**0.5. A new look at  $R$ -completion.** The three theorems above, taken together, provide us with a new way of looking at the concept of  $R$ -completion: more precisely, we have three notions associated to any ring  $R$ :

- (a) An  $\Theta_R$ -mapping algebra  $\mathfrak{X}$ , with the accompanying algebraic notion of the  $\Theta_R$ -algebra  $\Lambda = \pi_0 \mathfrak{X}$ ;
- (b) A cosimplicial weak  $\mathcal{G}$ -resolution  $\mathbf{W}^{\bullet}$  of the  $\Theta_R$ -mapping algebra  $\mathfrak{X}$ , for  $\mathcal{G} = \text{Obj } \Theta_R$ , with the accompanying algebraic notion of a free simplicial  $\Theta_R$ -algebra resolution  $V_{\bullet}$  of  $\pi_0 \mathfrak{X}$ ;
- (c) The realization  $\text{Tot } \mathbf{W}^{\bullet}$  (cf. §3.32) of the weak  $\mathcal{G}$ -resolution  $\mathbf{W}^{\bullet}$ , which is  $R$ -complete in the simply-connected case.

When  $R = \mathbb{F}_p$  or  $\mathbb{Q}$ , under mild assumptions on  $\mathfrak{X}$  (e.g., when it is simply connected and of finite type) we show that these notions are equivalent, inasmuch as the original  $\mathfrak{X}$  is in fact the  $\Theta_R$ -mapping algebra of  $\mathrm{Tot} \mathbf{W}^\bullet$ , up to weak equivalence.

Moreover, each of these three notions has certain advantages:

- (a) The  $\Theta_R$ -mapping algebra  $\mathfrak{X}$  exhibits in an explicit form all the information about a space  $\mathbf{Y}$  which is retained by its  $R$ -completion.
- (b) The cosimplicial space  $\mathbf{W}^\bullet$ , realizing the algebraic resolution  $V_\bullet$  of  $H^*(\mathbf{Y}; R)$ , encodes the higher order cohomology operations for  $\mathbf{Y}$  in a visible manner (see §4.32 below)
- (c) The  $R$ -complete space  $\mathrm{Tot} \mathbf{W}^\bullet = R_\infty \mathbf{Y}$  allows us to work with a single space which retains all the above information.

**0.6. Notation.** The category of topological spaces will be denoted by  $\mathcal{T}$ , and that of pointed topological spaces by  $\mathcal{T}_*$ . The category of simplicial sets will be denoted by  $\mathcal{S} = s\mathbf{Set}$ , that of pointed simplicial sets by  $\mathcal{S}_* = s\mathbf{Set}_*$ , and that of pointed Kan complexes by  $\mathcal{S}_*^{\mathrm{Kan}}$ .

Unless otherwise stated,  $\mathcal{C}$  will be a left proper pointed simplicial model category (cf. [Q, II, §2]), such as  $\mathcal{T}_*$ , and its objects will be denoted by boldface letters:  $\mathbf{X}, \mathbf{Y}, \dots$

**0.7. Organization.** In Section 1 we recall some facts about (co)simplicial objects, and in particular Bousfield's resolution model category structure. In Section 2 we define the notions of enriched sketches and mapping algebras, and in Section 3 we explain how this structure can be used to recover  $\mathbf{Y}$  from  $X = \mathrm{map}_*(\mathbf{Y}, \mathbf{A})$ , for suitable  $\mathbf{A}$ , proving Theorem A. In Section 4 we prove our main technical result, Theorem B, showing how any algebraic resolution of  $\pi_0 \mathfrak{X}$  in  $\mathcal{C}$ . This is used in Section 5 to prove Theorem C, which allows us to recognize mapping spaces of the form  $\mathrm{map}_*(\mathbf{Y}, \mathbf{K}(R, n))$  and recover  $\mathbf{Y}$  up to  $R$ -completion.

**0.8. Acknowledgements.** We wish to thank Pete Bousfield for many useful comments and elucidations of his work, and Paul Goerss for a helpful pointer.

## 1. THE RESOLUTION MODEL CATEGORY OF COSIMPLICIAL OBJECTS

The main technical tool for reconstructing the source of a mapping space  $X := \mathrm{map}_{\mathcal{C}}(\mathbf{Y}, \mathbf{A})$  in a pointed simplicial model category  $\mathcal{C}$ , such as  $\mathcal{T}_*$ , is the construction of a suitable cosimplicial resolution  $\mathbf{W}^\bullet$  of the putative  $\mathbf{Y} \in \mathcal{C}$ .

The proper framework for obtaining such a  $\mathbf{W}^\bullet$  is Bousfield's resolution model category of cosimplicial objects over  $\mathcal{C}$ , which generalizes and dualizes the Dwyer-Kan-Stover theory of the  $E_2$ -model category of simplicial spaces (cf. [DKSt]).

**1.1. Simplicial and cosimplicial objects.** We first collect some standard facts and constructions related to (co)simplicial objects in any category  $\mathcal{C}$ :

Let  $\Delta$  denote the category of finite ordered sets and order-preserving maps (cf. [M1, §2]), and  $\Delta^+$  the subcategory with the same objects, but only monic maps. A *cosimplicial object*  $G^\bullet$  in a category  $\mathcal{C}$  is a functor  $\Delta \rightarrow \mathcal{C}$ , and a *restricted cosimplicial object* is a functor  $\Delta^+ \rightarrow \mathcal{C}$ . More concretely, we write  $G^n$  for the value of  $G^\bullet$  at the ordered set  $[\mathbf{n}] = (0 < 1 < \dots < n)$ . The maps in the diagram  $G^\bullet$  are generated by the *coface* maps  $d^i = d_n^i : G^n \rightarrow G^{n+1}$  ( $0 \leq i \leq n+1$ ), as well as *codegeneracy* maps  $s^j = s_n^j : G^n \rightarrow G^{n-1}$  ( $0 \leq j < n$ ) in the non-restricted case, satisfying the usual cosimplicial identities.

Dually, a *simplicial object*  $G_\bullet$  in  $\mathcal{C}$  is a functor  $\Delta^{\text{op}} \rightarrow \mathcal{C}$ . The category  $\mathcal{C}^\Delta$  of cosimplicial objects over  $\mathcal{C}$  will be denoted by  $c\mathcal{C}$ , and the category  $\mathcal{C}^{\Delta^{\text{op}}}$  of simplicial objects over  $\mathcal{C}$  will be denoted by  $s\mathcal{C}$ .

There are natural embeddings  $c(-)^\bullet : \mathcal{C} \rightarrow c\mathcal{C}$  and  $c(-)_\bullet : \mathcal{C} \rightarrow s\mathcal{C}$ , defined by letting  $c(A)^\bullet$  denote the constant cosimplicial object which is  $A$  in every cosimplicial dimension, and similarly for  $c(A)_\bullet$ .

**1.2. Latching and matching objects.** For a cosimplicial object  $G^\bullet \in c\mathcal{C}$  in a (co)complete category  $\mathcal{C}$ , the  $n$ -th *matching object* for  $G^\bullet$  is defined to be

$$(1.3) \quad M^n G^\bullet := \lim_{\phi: [\mathbf{n}] \rightarrow [\mathbf{k}]} G^k,$$

where  $\phi$  ranges over the surjective maps  $[\mathbf{n}] \rightarrow [\mathbf{k}]$  in  $\Delta$ . There is a natural map  $\zeta^n : G^n \rightarrow M^n G^\bullet$  induced by the structure maps of the limit, and any iterated codegeneracy map  $s^I = \phi_* : G^k \rightarrow G^n$  factors as

$$(1.4) \quad s^I = \text{proj}_\phi \circ \zeta^n,$$

where  $\text{proj}_\phi : M^n G^\bullet \rightarrow G^k$  is the structure map for the copy of  $G^k$  indexed by  $\phi$  (cf. [BK1, X, §4.5]).

Similarly, the  $n$ -th *latching object* for  $G^\bullet \in c\mathcal{C}$  is the colimit

$$(1.5) \quad L^n G^\bullet := \text{colim}_{\theta: [\mathbf{k}] \rightarrow [\mathbf{n}]} G^k,$$

where  $\theta$  ranges over the injective maps  $[\mathbf{k}] \rightarrow [\mathbf{n}]$  in  $\Delta$  (for  $k < n$ ), with  $\sigma^n : L^n G^\bullet \rightarrow G^n$  defined by the structure maps of the colimit.

These two constructions have analogues for a simplicial object  $G_\bullet$  over a (co)complete category  $\mathcal{C}$ : the *latching object*  $L_n G_\bullet := \text{colim}_{\theta: [\mathbf{k}] \rightarrow [\mathbf{n}]} G_k$ , and the *matching object*  $M_n G_\bullet := \lim_{\phi: [\mathbf{n}] \rightarrow [\mathbf{k}]} G_k$ , equipped with the obvious canonical maps.

**1.6. Definition.** If  $\mathcal{C}$  is pointed and complete, the  $n$ -th *Moore chains* object of  $G_\bullet \in s\mathcal{C}$  is defined to be:

$$(1.7) \quad C_n G_\bullet := \bigcap_{i=1}^n \text{Ker}\{d_i : G_n \rightarrow G_{n-1}\},$$

with differential  $\partial_n^{G_\bullet} = \partial_n := (d_0)|_{C_n G_\bullet} : C_n G_\bullet \rightarrow C_{n-1} G_\bullet$ . The  $n$ -th *Moore cycles* object is  $Z_n G_\bullet := \text{Ker}(\partial_n^{G_\bullet})$ .

Dually, if  $\mathcal{C}$  is pointed and cocomplete, the  $n$ -th *Moore cochains* object of  $G^\bullet \in c\mathcal{C}$ , written  $C^n G^\bullet$ , is defined to be the colimit of:

$$(1.8) \quad \begin{array}{ccc} \prod_{i=1}^n G^{n-1} & \xrightarrow{\perp_i d^i} & G^n \\ \downarrow & & \\ & & * \end{array}$$

with differential  $\delta^n : C^{n-1} G^\bullet \rightarrow C^n G^\bullet$  induced by  $d^0$ .

**1.9. Definition.** A simplicial object  $G_\bullet \in s\mathcal{C}$  over a pointed category  $\mathcal{C}$  is called a *CW object* if it is equipped with a *CW basis*  $(\overline{G}_n)_{n=0}^\infty$  in  $\mathcal{C}$  such that  $G_n = \overline{G}_n \amalg L_n G_\bullet$ , and  $d_i|_{\overline{G}_n} = 0$  for  $1 \leq i \leq n$ . In this case  $\overline{\partial}_0^{G_n} := d_0|_{\overline{G}_n} : \overline{G}_n \rightarrow G_{n-1}$  is called the attaching map for  $\overline{G}_n$ . By the simplicial identities  $\overline{\partial}_0^{G_n}$  factors as

$$(1.10) \quad \overline{\partial}_0^{G_n} : \overline{G}_n \rightarrow Z_{n-1} G_\bullet \subset G_{n-1}.$$

In this case we have an explicit description

$$(1.11) \quad L_n G_\bullet := \coprod_{0 \leq k \leq n} \coprod_{0 \leq i_1 < \dots < i_{n-k-1} \leq n-1} \overline{G}_k$$

for its  $n$ -th latching object, in which the iterated degeneracy map  $s_{i_{n-k-1}} \dots s_{i_2} s_{i_1}$ , restricted to the basis  $\overline{G}_k$ , is the inclusion into the copy of  $\overline{G}_k$  indexed by  $(i_1, \dots, i_{n-k-1})$ .

A cosimplicial CW object may be defined analogously, but we shall only need the following variant:

**1.12. Definition.** A cosimplicial pointed space  $\mathbf{W}^\bullet$  equipped with a CW basis  $\overline{\mathbf{W}}^n$  ( $n \geq 0$ ) in  $\mathcal{T}_*$  is called a *weak CW object* if

- (a) For each  $n \geq 0$ , we have a weak equivalence  $\varphi^n : \mathbf{W}^n \xrightarrow{\sim} \overline{\mathbf{W}}^n \times M^n \mathbf{W}^\bullet$ , and we set

$$(1.13) \quad \overline{\varphi}^n := \text{proj}_{\overline{\mathbf{W}}^n} \circ \varphi^n : \mathbf{W}^n \rightarrow \overline{\mathbf{W}}^n.$$

where  $\text{proj}_{\overline{\mathbf{W}}^n} : \mathbf{W}^n \rightarrow \overline{\mathbf{W}}^n$  is the projection.

- (b)  $\overline{\varphi}^n \circ d_{n-1}^i \sim 0$  for  $1 \leq i \leq n$ .

- (c) If we define the *attaching map* for  $\overline{\mathbf{W}}^n$  to be  $\overline{d}_{n-1}^0 := \overline{\varphi}^{n-1} \circ d_{n-1}^0 : \mathbf{W}^{n-1} \rightarrow \overline{\mathbf{W}}^n$ , we require that it be a “Moore cochain” in the sense that

$$(1.14) \quad \overline{d}_{n-1}^0 \circ d_{n-2}^i = 0$$

for all  $1 \leq i \leq n-2$ .

**1.15. Remark.** Recall that a *simplicial* model category  $\mathcal{C}$  is one in which, for each (finite)  $K \in \mathcal{S}$  and  $X \in \mathcal{C}$ , we have objects  $X \otimes K$  and  $X^K$  in  $\mathcal{C}$  equipped with appropriate adjunction-like isomorphisms and axiom SM7 (see [Q, II, §1-2]). In particular, such model categories are simplicially enriched.

**1.16. Reedy model structure.** If  $\mathcal{C}$  is a model category, the Reedy model structure on  $c\mathcal{C}$  (cf. [Bo1, §2.2]) is defined by letting a cosimplicial map  $f : \mathbf{X}^\bullet \rightarrow \mathbf{Y}^\bullet$  in  $c\mathcal{C}$  be:

- (i) a *Reedy weak equivalence* when  $f : X^n \rightarrow Y^n$  is a weak equivalence in  $\mathcal{C}$  for  $n \geq 0$ ;
- (ii) a *Reedy cofibration* when  $X^n \coprod_{L^n \mathbf{X}^\bullet} L^n \mathbf{Y}^\bullet \rightarrow Y^n$  is a cofibration in  $\mathcal{C}$  for  $n \geq 0$ ;
- (iii) a *Reedy fibration* when  $X^n \rightarrow Y^n \prod_{M^n \mathbf{X}^\bullet} M^n \mathbf{X}^\bullet$  is a fibration in  $\mathcal{C}$  for  $n \geq 0$ .

The Reedy model category structure on  $s\mathcal{C}$  is defined dually (see [Hi, §15.3]).

**1.17.  $\mathcal{G}$ -resolution model structure.** Let  $\mathcal{G}$  be a class of homotopy group objects in a pointed model category  $\mathcal{C}$ , closed under loops. A map  $i : A \rightarrow B$  in  $\text{ho}\mathcal{C}$  is called  *$\mathcal{G}$ -monic* if  $i^* : [B, G] \rightarrow [A, G]$  is onto for each  $G \in \mathcal{G}$ . An object  $Y$  in  $\mathcal{C}$  is called  *$\mathcal{G}$ -injective* if  $i^* : [B, Y] \rightarrow [A, Y]$  is onto for each  $\mathcal{G}$ -monic map  $i : A \rightarrow B$  in  $\text{ho}\mathcal{C}$ . A fibration in  $\mathcal{C}$  is called  *$\mathcal{G}$ -injective* if it has the right lifting property for the  $\mathcal{G}$ -monic cofibrations in  $\mathcal{C}$ .

The homotopy category  $\text{ho}\mathcal{C}$  is said to have *enough  $\mathcal{G}$ -injectives* if each object is the source of a  $\mathcal{G}$ -monic map to a  $\mathcal{G}$ -injective target. In this case  $\mathcal{G}$  is called a class of *injective models* in  $\text{ho}\mathcal{C}$ .

Recall that a homomorphism in the category  $s\mathcal{G}p$  of simplicial groups is a weak equivalence or fibration when its underlying map in  $\mathcal{S}$  is such. A map  $f : \mathbf{X}^\bullet \rightarrow \mathbf{Y}^\bullet$  in  $c\mathcal{C}$  is called

- (i) a  $\mathcal{G}$ -equivalence if  $f^* : [\mathbf{Y}^\bullet, G] \rightarrow [\mathbf{X}^\bullet, G]$  is a weak equivalence in  $s\mathcal{G}p$  for each  $G \in \mathcal{G}$ ;
- (ii) a  $\mathcal{G}$ -cofibration if  $f$  is a Reedy cofibration and  $f^* : [\mathbf{Y}^\bullet, G] \rightarrow [\mathbf{X}^\bullet, G]$  is a fibration in  $s\mathcal{G}p$  for each  $G \in \mathcal{G}$ ;
- (iii) a  $\mathcal{G}$ -fibration if  $f : X^n \rightarrow Y^n \times_{M^n \mathbf{Y}^\bullet} M^n \mathbf{X}^\bullet$  is a  $\mathcal{G}$ -injective fibration in  $\mathcal{C}$  for each  $n \geq 0$ .

In [Bo1, Theorem 3.3], Bousfield showed that if  $\mathcal{C}$  is a left proper pointed model category and  $\mathcal{G}$  is a class of injective models in  $\text{ho}(\mathcal{C})$ , the above defines a left proper pointed simplicial model category structure on  $c\mathcal{C}$ .

**1.18. Definition.** Given a class  $\mathcal{G}$  of homotopy group objects in a model category  $\mathcal{C}$  as above, a cosimplicial object  $\mathbf{W}^\bullet \in c\mathcal{C}$  is called *weakly  $\mathcal{G}$ -fibrant* if it is Reedy fibrant, and every  $W^n$  is in  $\mathcal{G}$  ( $n \geq 0$ ). A *weak  $\mathcal{G}$ -resolution* of an object  $Y \in \mathcal{C}$  is a weakly  $\mathcal{G}$ -fibrant  $\mathbf{W}^\bullet$  which is  $\mathcal{G}$ -equivalent to  $c(Y)^\bullet$  (cf. §0.6) See [Bo1, §6].

## 2. ENRICHED SKETCHES AND MAPPING ALGEBRAS

We now set up the categorical framework needed to describe the relevant extra structure on a mapping space.

**2.1. Definition.** Let  $\Theta$  be an *sketch*, in the sense of Ehresmann (cf. [E], [Bor, §5.6]): that is, a small pointed category with a distinguished set  $\mathcal{P}$  of (small) products (including the empty product  $*$ ). A  $\Theta$ -algebra is a functor  $\Lambda : \Theta \rightarrow \mathbf{Set}_*$  which preserves the products in  $\mathcal{P}$ . We think of a map  $\phi : \prod_{i < \kappa} a_i \rightarrow \prod_{j < \lambda} b_j$  in  $\Theta$  as representing an  $\lambda$ -valued  $\kappa$ -ary operation on  $\Theta$ -algebras, with gradings indexed by  $(a_i)_{i < \kappa}$  and  $(b_j)_{j < \lambda}$ , respectively.

The category of  $\Theta$ -algebras is denoted by  $\Theta\text{-Alg}$ . If each object of  $\Theta$  is uniquely representable (up to order) as a product of elements in a set  $\mathcal{O} \subseteq \text{Obj } \Theta$ , there is a forgetful functor  $U : \Theta\text{-Alg} \rightarrow \mathbf{Set}^{\mathcal{O}}$  into the category of  $\mathcal{O}$ -graded sets, with left adjoint the *free  $\Theta$ -algebra* functor  $F : \mathbf{Set}^{\mathcal{O}} \rightarrow \Theta\text{-Alg}$ .

**2.2. Example.** The simplest kind of a sketch is a *theory* in the sense of Lawvere (cf. [L]), in which  $\text{Obj } \Theta = \mathbb{N}$  is generated under products by a single object, so that  $\Theta$ -algebras are simply sets with additional structure. For example, the theory  $\mathfrak{G}$  whose algebras are groups is just the opposite category of the homotopy category of finite wedges of circles.

**2.3. Definition.** We define a  $\mathfrak{G}$ -sketch to be a sketch  $\Theta$  equipped with an embedding of sketches  $\mathfrak{G}^{\mathcal{O}} \hookrightarrow \Theta$ , for  $\mathcal{O}$  as above. In this case, any  $\Theta$ -algebra  $\Lambda$  has a natural underlying  $\mathcal{O}$ -graded group structure. We do not require the operations of a  $\mathfrak{G}$ -sketch to be homomorphisms (that is, commute with the  $\mathfrak{G}$ -structure).

**2.4. Example.** Almost all varieties of (graded) universal algebras, in the sense of [Mc, V, §6] – such as groups, associative, or Lie algebras, and so on – have an underlying (graded) group structure, so they are categories of  $\Theta$ -algebras for a suitable  $\mathfrak{G}$ -sketch  $\Theta$ .

**2.5. Proposition.** *If  $\Theta$  is a  $\mathfrak{G}$ -sketch, the category  $s\Theta\text{-Alg}$  of simplicial  $\Theta$ -algebras has a model category structure, in which the weak equivalences and fibrations are defined objectwise.*

*Proof.* See [BP, §6], which is a slight generalization of [Q, II, §4].  $\square$

**2.6. Enriched sketches and algebras.** There is also an enriched version of the notions defined above, introduced in [BB1], in which we assume that the theory, or sketch, is simplicially enriched, and the algebras over it are simplicial. This takes place in the context of a simplicially enriched model category. Note that in fact any model category  $\mathcal{C}$  can be enriched over  $\mathcal{S}$  – that is, for each  $X, Y \in \mathcal{C}$ , there is a simplicial mapping space  $\text{map}_{\mathcal{C}}(X, Y)$ , with continuous compositions, such that  $[X, Y]_{\text{ho}\mathcal{C}}$  is equal to its set of components  $\pi_0 \text{map}_{\mathcal{C}}(X, Y)$  (cf. [DK1]).

**2.7. Definition.** Let  $\mathcal{C}$  be a simplicial model category as in §0.6, and  $\lambda$  some limit cardinal (to be determined by the context – see Remark 4.7 below; often,  $\lambda = \aleph_0$ ). An *enriched sketch*  $\Theta$  is a small full sub-simplicial category of  $\mathcal{C}$ , closed under loops (cf. [Q, I, §2]) and products of cardinality  $< \lambda$ . We assume all objects in  $\Theta$  are fibrant and cofibrant homotopy group objects in  $\mathcal{C}$ .

**2.8. Example.** Let  $\mathcal{A} = (\underline{\mathbf{A}}_n)_{n=0}^\infty$  be an  $\Omega$ -spectrum, so each  $\underline{\mathbf{A}}_n \cong \Omega \underline{\mathbf{A}}_{n+1}$  is an  $\Omega^\infty$ -space, and let  $\Theta_{\mathcal{A}}$  denote the full sub-simplicial category of  $\mathcal{C}$  whose objects are products of the spaces of  $\mathcal{A}$  of cardinality  $< \lambda$ .

In particular, for any abelian group  $R$  we let  $\mathcal{A} = (\mathbf{K}(R, n))_{n=0}^\infty$  be the corresponding Eilenberg-Mac Lane spectrum, and denote the resulting enriched sketch  $\Theta_{\mathcal{A}}$  by  $\Theta_R$ , whose objects are finite type  $R$ -GEMs (generalized Eilenberg-Mac Lane spaces) that is, spaces of the form  $\prod_{i=1}^N \mathbf{K}(R, m_i)$  ( $m_i \geq 1$ ).

Note that in this case we may assume that each object in  $\Theta_R$  is a strict abelian group object.

**2.9. Definition.** Recall that in a pointed simplicial model category  $\mathcal{C}$  one has “mapping objects”  $\mathbf{X}^K \in \mathcal{C}$  for any  $\mathbf{X} \in \mathcal{C}$  and finite simplicial set  $K \in \mathcal{S}$  (see [Q, II, §1]). In particular, the inclusions  $i_0, i_1 : * \hookrightarrow \Delta[1]$  induce natural “evaluation maps”  $\text{ev}_0, \text{ev}_1 : \mathbf{X}^{\Delta[1]} \rightarrow \mathbf{X}$ , which are trivial fibrations, for any  $\mathbf{X} \in \mathcal{C}$ . This allows one to define the *path* and *loop* objects in  $\mathcal{C}$  by the pullback diagrams:

$$(2.10) \quad \begin{array}{ccc} P\mathbf{X} & \xrightarrow{\quad} & \mathbf{X}^{\Delta[1]} \\ \simeq \downarrow & \boxed{\text{PB}} & \downarrow \text{ev}_0 \simeq \\ * & \xrightarrow{\quad} & \mathbf{X} \end{array} \quad \begin{array}{ccc} \Omega\mathbf{X} & \xrightarrow{\quad} & P\mathbf{X} \\ \downarrow & \boxed{\text{PB}} & \downarrow \text{ev}_1 \\ * & \xrightarrow{\quad} & \mathbf{X} \end{array}$$

These will also be our models for simplicial path and loop spaces  $PK$  and  $\Omega K$  for any Kan complex  $K \in \mathcal{S}_*^{\text{Kan}}$ .

**2.11. Definition.** For  $\mathcal{C}$  a model category as in §0.6, and  $\Theta \subseteq \mathcal{C}$  an enriched sketch as above, a  $\Theta$ -*mapping algebra* is a pointed simplicial functor  $\mathfrak{X} : \Theta \rightarrow \mathcal{S}_*$ , taking values in Kan complexes, satisfying the following three conditions:

- (a) The natural map  $\mathfrak{X}\{\prod_{i < \lambda} \mathbf{B}_i\} \rightarrow \prod_{i < \lambda} \mathfrak{X}\{\mathbf{B}_i\}$  is an isomorphism for all collections of  $\leq \lambda$  objects  $\mathbf{B}_i \in \Theta$  (where we write  $\mathfrak{X}\{\mathbf{B}\}$  for the value of  $\mathfrak{X}$  at  $\mathbf{B} \in \Theta$ ).
- (b) Using the convention that  $\mathfrak{X}\{\mathbf{B}^K\} := (\mathfrak{X}\{\mathbf{B}\})^K$ , for any finite simplicial set  $K$ , we require that  $\mathfrak{X}$  preserve all the pullback squares of the form (2.10), so we have natural identifications  $\mathfrak{X}\{P\mathbf{B}\} = P\mathfrak{X}\{\mathbf{B}\}$  and  $\mathfrak{X}\{\Omega\mathbf{B}\} = \Omega\mathfrak{X}\{\mathbf{B}\}$ .
- (c) Any cofibration  $i : \mathbf{B} \hookrightarrow \mathbf{B}'$  in  $\Theta$  induces an inclusion  $i_\# : \mathfrak{X}\{\mathbf{B}\} \hookrightarrow \mathfrak{X}\{\mathbf{B}'\}$  for all  $\mathbf{B} \in \Theta$ .

The category of  $\Theta$ -mapping algebras will be denoted by  $\text{Map}_\Theta$ .



**2.12. Example.** For a given object  $\mathbf{Y} \in \mathcal{C}$ , we have a *realizable*  $\Theta$ -mapping algebra  $\mathfrak{M}_\Theta \mathbf{Y}$  defined for any  $\mathbf{B} \in \Theta$  by  $\mathfrak{M}_\Theta \mathbf{Y}\{\mathbf{B}\} := \text{map}_{\mathcal{C}}(\mathbf{Y}, \mathbf{B})$ . When  $\mathcal{C} = \mathcal{T}_*$  and  $\Theta = \Theta_R$ , we shall denote this by  $\mathfrak{M}_R \mathbf{Y}$ . The realizable  $\Theta$ -mapping algebra  $\mathfrak{M}_\Theta \mathbf{B}$  for  $\mathbf{B} \in \Theta$  will be called *free*.

We then have:

**2.13. Lemma** (cf. [BB1, 8.17]). *If  $\mathfrak{Y}$  is an  $\Theta$ -mapping algebra and  $\mathfrak{M}_\Theta \mathbf{B}$  is a free  $\Theta$ -mapping algebra (for  $\mathbf{B} \in \Theta$ ), there is a natural isomorphism*

$$\Phi : \text{map}_{\text{Map}_\Theta}(\mathfrak{M}_\Theta \mathbf{B}, \mathfrak{Y}) \xrightarrow{\cong} \mathfrak{Y}\{\mathbf{B}\},$$

with  $\Phi(f) = f(\text{Id}_{\mathbf{B}}) \in \mathfrak{Y}\{\mathbf{B}\}_0$  for any  $f \in \text{Hom}_{\text{Map}_\Theta}(\mathfrak{M}_\Theta \mathbf{B}, \mathfrak{Y}) = \text{map}_{\text{Map}_\Theta}(\mathfrak{M}_\Theta \mathbf{B}, \mathfrak{Y})_0$ .

*Proof.* This follows from the strong Yoneda Lemma for enriched categories (see [Ke, 2.4]).  $\square$

**2.14. Definition.** Given a homotopy group object  $\mathbf{A}$  in a model category  $\mathcal{C}$  as in §0.6, we have an associated enriched sketch  $\Theta = \Theta_{\mathbf{A}}$ , whose objects are of the form  $\mathbf{B} = \prod_{i \in I_0} \Omega^{n_i} \mathbf{A}$  with  $|I| < \lambda$ . In this case a  $\Theta$ -mapping algebra structure on a simplicial set  $X$  is a  $\Theta$ -mapping algebra  $\mathfrak{X}$  with  $\mathfrak{X}\{\mathbf{A}\} = X$  (so  $\mathfrak{X}\{\prod_{i \in I_0} \Omega^{n_i} \mathbf{A}\} = \prod_{i \in I} \Omega^{n_i} X$ ).

Similarly, if  $\mathcal{A} = (\underline{\mathbf{A}}_n)_{n \in \mathbf{Z}}$  is an  $\Omega$ -spectrum in  $\mathcal{C}$ , as in §2.8, an  $\Theta_{\mathcal{A}}$ -mapping algebra structure on  $X$  is an  $\Theta_{\mathcal{A}}$ -mapping algebra  $\mathfrak{X}$  such that  $\mathfrak{X}(\prod_{i \in I} \underline{\mathbf{A}}_{n_i}) = \prod_{i \in I} \Omega^{-n_i} X$  – which implicitly involves choices of deloopings of  $X$ .

**2.15. Remark.** The  $\Theta$ -mapping algebra structures we define here are rigid, in the sense that the action of the mapping spaces between objects of  $\Theta$  is strict. In particular, the fact that  $X = \text{map}_{\mathcal{C}}(\mathbf{Y}, \mathbf{A})$  has an  $\Theta$ -mapping algebra structure (for  $\Theta = \Theta_{\mathbf{A}}$  as above) has no implications for any  $X' \simeq X$ . This defect can be remedied by defining a suitable notion of a *lax*  $\Theta$ -mapping algebra, as was done in the dual case in [BBD, §6].

**2.16. The associated “algebraic” sketch.** To any enriched sketch  $\Theta$  in a simplicial model category  $\mathcal{C}$  we can associate a  $\mathfrak{S}$ -sketch  $\Theta := \pi_0 \Theta$ , with the same objects as  $\Theta$ , where  $\text{Hom}_\Theta(A, B) := \pi_0 \text{map}_\Theta(A, B)$ .

A  $\Theta$ -algebra  $\Lambda : \pi_0 \Theta \rightarrow \mathbf{Set}_*$  is called *enrichable* if it is of the form  $\Lambda_{\mathfrak{X}} := \pi_0 \mathfrak{X}$  for some  $\mathfrak{X} \in \text{Map}_\Theta$  (not necessarily unique).

We define a map of  $\Theta$ -mapping algebras  $f : \mathfrak{X} \rightarrow \mathfrak{Y}$  to be a *weak equivalence* if it induces an isomorphism  $f_\# : \pi_0 \mathfrak{X} \rightarrow \pi_0 \mathfrak{Y}$  of the corresponding  $\Theta$ -algebras.

**2.17. Definition.** A  $\Theta$ -algebra  $\Lambda : \pi_0 \Theta \rightarrow \mathbf{Set}_*$  is *realizable* (in  $\mathcal{C}$ ) if it is enrichable by a *realizable*  $\Theta$ -mapping algebra  $\mathfrak{M}_\Theta \mathbf{Y}$  – that is,  $\Lambda \cong \Lambda_{\mathfrak{M}_\Theta \mathbf{Y}}$  for some  $\mathbf{Y} \in \mathcal{C}$  (again, not necessarily unique). In this case we say that  $\mathbf{Y}$  *realizes*  $\Lambda$ . Any  $\Theta$ -algebra of the form  $\pi_0 \mathfrak{X}$ , where  $\mathfrak{X}$  is a free  $\Theta$ -mapping algebra, will be called a *free  $\Theta$ -algebra*.

**2.18. Remark.** In principle, any coproduct of free  $\Theta$ -mapping algebras or  $\Theta$ -algebras is also free (in the sense of being in the image of the left adjoint of an appropriate forgetful functor). However, in order to avoid the question of realizability for arbitrary free  $\Theta$ -mapping algebras (or  $\Theta$ -algebras), we restrict attention to coproducts of monogenic objects of cardinality  $< \lambda$ .

We also have an algebraic version of Lemma 2.13, which follows from the usual Yoneda Lemma:

**2.19. Lemma.** *If  $\Lambda$  is an  $\Theta$ -algebra and  $\pi_0 \mathfrak{M}_\Theta \mathbf{B}$  is a free  $\Theta$ -algebra (for  $\mathbf{B} \in \Theta$ ), there is a natural isomorphism  $\mathrm{Hom}_{\Theta\text{-Alg}}(\pi_0 \mathfrak{M}_\Theta \mathbf{B}, \Lambda) \cong \Lambda\{\mathbf{B}\}$ .*

**2.20. Example.** For any abelian group  $R$ , with  $\Theta_R$  as in §2.8, we obtain an  $\mathfrak{G}$  sketch  $\Theta_R := \pi_0 \Theta_R$ , namely, the full subcategory of  $\mathrm{ho} \mathcal{T}_*$  whose objects are finite type  $R$ -GEMs (which are abelian group objects in  $\mathrm{ho} \mathcal{T}$ ). Note that the cohomology functor  $H^*(-; R)$  in fact lands in  $\Theta_R\text{-Alg}$ , and realizable  $\Theta_R$ -algebras are those which correspond to actual spaces.

**2.21. Simplicial  $\Theta$ -mapping algebras.** Unfortunately, there seems to be no useful model category structure on the category  $\mathrm{Map}_\Theta$  of  $\Theta$ -mapping algebras. However, we do have a model category structure on the functor category  $\mathcal{S}_*^\Theta$ , in which weak equivalences and fibrations are defined objectwise (cf. [DK2, §7], and compare [BB1, §8]), and any free  $\Theta$ -mapping algebra is in fact a homotopy cogroup object for  $\mathcal{S}_*^\Theta$ . Thus we obtain a resolution model category structure (cf. [Bo1, J]) on the category  $s\mathcal{S}_*^\Theta$  of *simplicial* simplicially-enriched functors  $\Theta \rightarrow \mathcal{S}_*$ , in which a map  $f : \mathfrak{Y}_\bullet \rightarrow \mathfrak{W}_\bullet$  is a *weak equivalence* if for each  $\mathbf{B} \in \Theta$ , the map  $\pi_0 \mathfrak{Y}_\bullet\{\mathbf{B}\} \rightarrow \pi_0 \mathfrak{W}_\bullet\{\mathbf{B}\}$  is a weak equivalence of simplicial groups (see [BB2, §2.2]).

### 3. MAPPING ALGEBRAS AND COMPLETIONS

We now explain how the  $\Theta_{\mathcal{A}}$ -mapping algebra structure on  $X = \mathrm{map}_*(\mathbf{Y}, \mathbf{A})$  described in §2.14 suffices to recover  $\mathbf{Y}$  from  $X$ , up to a suitable notion of  $\mathcal{A}$ -weak equivalence, generalizing the results of [BB2, §3] for  $\Theta = \Theta_R$ .

**3.1. Remark.** Our results apply more generally to any enriched theory  $\Theta$  and realizable  $\Theta$ -mapping algebra  $\mathfrak{X}$ , but the formulation is slightly more complicated, so for simplicity we restrict attention to the case  $\Theta = \Theta_{\mathcal{A}}$ .

In fact, the procedure we describe in this section actually works for *any*  $\Theta_{\mathcal{A}}$ -mapping algebra  $\mathfrak{X}$ , whether or not it is realizable, and yields a  $\mathcal{G}$ -complete space  $\hat{\mathbf{Y}}$  for  $\mathcal{G} = \mathrm{Obj} \Theta_{\mathcal{A}}$ , which we can think of as the  $\mathcal{G}$ -completion of the  $\Theta_{\mathcal{A}}$ -mapping algebra  $\mathfrak{X}$ . We therefore denote  $\hat{\mathbf{Y}}$  by  $\hat{L}_{\mathcal{G}} \mathfrak{X}$  (compare [Bo1, §5.7]). However, in general we cannot show that  $\mathfrak{M}_{\mathcal{A}} \hat{\mathbf{Y}}$  is weakly equivalent to the given  $\mathfrak{X}$  – unless  $\mathfrak{X} = \mathfrak{M}_{\mathcal{A}} \mathbf{Y}$  to begin with, in which case we shall see that  $\hat{\mathbf{Y}} \simeq \hat{L}_{\mathcal{G}} \mathbf{Y}$ , so that  $\hat{L}_{\mathcal{G}} \mathfrak{M}_{\Theta} \mathbf{Y} \simeq \mathfrak{M}_{\mathcal{A}} \hat{L}_{\mathcal{G}} \mathbf{Y}$ .

By construction we have a functor  $\pi_0 : \mathrm{Map}_{\mathcal{A}} \rightarrow \Theta_{\mathcal{A}}\text{-Alg}$  associating to any  $\Theta$ -mapping algebra  $\mathfrak{X}$  its  $\Theta$ -algebra  $\pi_0 \mathfrak{X}$  (cf. §2.16). Since  $\Theta_{\mathcal{A}}$ -mapping algebras are rather complicated objects, it is natural to ask whether this functor factors through some simpler category. Evidently, for any fibrant simplicial set  $K$ ,  $\pi_0 K$  depends only on the 0-simplices and their homotopies, i.e., on the 1-truncation  $\tau_1 K$  of  $K$ . However, we need even less information if  $K$  is a group object in  $\mathrm{ho} \mathcal{S}_*$ :

**3.2. Definition.** In model category  $\mathcal{C}$  (as in §0.6), an *H-group* is a fibrant and cofibrant homotopy group object – that is, an object  $\mathbf{X} \in \mathcal{C}$  equipped with structure maps  $\mu : \mathbf{X} \times \mathbf{X} \rightarrow \mathbf{X}$  and  $(-)^{-1} \mathbf{X} \rightarrow \mathbf{X}$ , (with  $* \hookrightarrow \mathbf{X}$  as the identity element), as well as chosen homotopies for each of the identities in  $\mathfrak{G}$  (cf. §2.2) such as:  $H : \mu \circ (\mu \times \mathrm{Id}) \sim \mu \circ (\mathrm{Id} \times \mu)$  for the associativity,  $G : \mu \circ ((-)^{-1} \times \mathrm{Id}) \circ \mathrm{diag} \sim c_*$  for the left inverse, and so on.

For any cofibrant  $\mathbf{Y} \in \mathcal{C}$ , the Kan complex  $K := \mathrm{map}_{\mathcal{C}}(\mathbf{Y}, \mathbf{X})$  inherits an *H-group* structure in  $\mathcal{S}_*$ . We define an equivalence relation on its 0-simplices by

setting:

$$(3.3) \quad f \sim g \iff \exists \alpha \in K_1 \text{ such that } d_0 \alpha = \mu_*(f, g^{-1}) \text{ and } d_1 \alpha = * .$$

for  $f, g \in K_0$ . Note that  $\alpha \in (PK)_0$  (see §2.9).

**3.4. Lemma.** *For any  $H$ -group  $\mathbf{X} \in \mathcal{C}$  as above,*

$$\pi_0 \text{map}_{\mathcal{C}}(\mathbf{Y}, \mathbf{X}) \cong (\text{map}_{\mathcal{C}}(\mathbf{Y}, \mathbf{X}))_0 / \sim .$$

**3.5. Definition.** If  $\text{Obj } \mathcal{A}$  is the set of spaces of an  $\Omega$ -spectrum  $\mathcal{A} = (\underline{\mathbf{A}}_n)_{n \in \mathbb{Z}}$  in a pointed model category  $\mathcal{C}$ , a *discrete  $\mathcal{A}$ -mapping algebra* is a function  $\mathfrak{Y} : \text{Obj } \mathcal{A} \rightarrow \mathbf{Set}_*^J$ , written  $\underline{\mathbf{A}}_n \mapsto (P\mathfrak{Y}\{\underline{\mathbf{A}}_n\} \xrightarrow{p_n} \mathfrak{Y}\{\underline{\mathbf{A}}_n\})$  ( $n \in \mathbb{Z}$ ), where  $J$  denotes the single-arrow category  $0 \rightarrow 1$ . The category of discrete  $\mathcal{A}$ -mapping algebras will be denoted by  $\text{Map}_{\mathcal{A}, \text{d}}$ .

**3.6. Example.** Let  $\rho : \mathcal{S}_*^{\text{Kan}} \rightarrow \mathbf{Set}_*^J$  be the functor assigning to a Kan complex  $K$  the 0-simplices  $p_0 : (PK)_0 \rightarrow K_0$  of its path fibration  $p : PK \rightarrow K$  (cf. §2.9).

If  $\mathfrak{X}$  is an  $\Theta_{\mathcal{A}}$ -mapping algebra for  $\mathcal{A}$  as above, the associated discrete  $\mathcal{A}$ -mapping algebra  $\rho\mathfrak{X}$  is defined by setting  $\rho\mathfrak{X}\{\underline{\mathbf{A}}_n\} := p_0 : \mathfrak{X}\{P\underline{\mathbf{A}}_n\}_0 \rightarrow \mathfrak{X}\{\underline{\mathbf{A}}_n\}_0$ . This defines a functor  $\rho : \text{Map}_{\mathcal{A}} \rightarrow \text{Map}_{\mathcal{A}, \text{d}}$ , since  $\mathfrak{X}$  takes values in Kan complexes. Moreover, we may define a covariant functor  $\mathcal{L}_{\mathcal{A}} : \mathcal{C} \rightarrow \text{Map}_{\mathcal{A}, \text{d}}^{\text{op}}$  by  $(\mathcal{L}_{\mathcal{A}}\mathbf{Y})\{\underline{\mathbf{A}}_n\} := \rho\mathfrak{M}_{\mathcal{A}}\mathbf{Y}\{\underline{\mathbf{A}}_n\}$ .

**3.7. Remark.** Note that  $\text{Map}_{\mathcal{A}, \text{d}}$  is just the diagram category  $\mathbf{Set}_*^{\Gamma}$ , indexed by a linear category  $\Gamma$  consisting of a single non-identity arrow  $q_n : P_n \rightarrow A_n$  for each  $n \in \mathbb{Z}$ , and thus no non-trivial compositions.

Thus we have a pullback diagram:

$$(3.8) \quad \begin{array}{ccc} \text{Hom}_{\mathbf{Set}_*^{\Gamma}}(\mathfrak{X}, \mathfrak{Y}) & \xrightarrow{\quad} & \prod_{n \in \mathbb{Z}} \text{Hom}_{\mathbf{Set}_*}(\mathfrak{X}\{P_n\}, \mathfrak{Y}\{P_n\}) \\ \downarrow & \boxed{\text{PB}} & \downarrow \mathfrak{Y}\{q_n\}_* \\ \prod_{n \in \mathbb{Z}} \text{Hom}_{\mathbf{Set}_*}(\mathfrak{X}\{A_n\}, \mathfrak{Y}\{A_n\}) & \xrightarrow{\mathfrak{X}\{q_n\}^*} & \prod_{n \in \mathbb{Z}} \text{Hom}_{\mathbf{Set}_*}(\mathfrak{X}\{P_n\}, \mathfrak{Y}\{A_n\}) \end{array}$$

for any  $\mathfrak{X}, \mathfrak{Y} \in \mathbf{Set}_*^{\Gamma}$ . Therefore,  $\text{Hom}_{\mathbf{Set}_*^{\Gamma}}(\mathfrak{X}, \mathfrak{Y})$  is a product over  $n \in \mathbb{Z}$  (of pullback squares).

We deduce from Lemma 3.4 and the fact that each  $\underline{\mathbf{A}}_n$  in  $\mathcal{A}$  is an (infinite) loop space:

**3.9. Lemma.** *For any  $\Theta_{\mathcal{A}}$ -mapping algebra  $\mathfrak{X}$ , the  $\Theta_{\mathcal{A}}$ -algebra  $\pi_0\mathfrak{X}$  is determined by  $\rho\mathfrak{X}$  (together with the maps  $\mu_* : \mathfrak{X}\{\underline{\mathbf{A}}_n \times \underline{\mathbf{A}}_n\}_0 \rightarrow \rho\mathfrak{X}\{\underline{\mathbf{A}}_n\}_0$ ).*

**3.10. The dual Stover construction.** In [Sto], Stover described a certain comonad on topological spaces which eventually was shown to produce simplicial resolutions in the  $E^2$ -model category of [DKSt]. We now give a more conceptual description of the dual construction.

Since the functor  $\mathcal{L}_{\mathcal{A}} : \mathcal{T}_* \rightarrow \text{Map}_{\mathcal{A}, \text{d}}^{\text{op}}$  of §3.6 has sufficient information to calculate the homotopy groups  $\pi_0(\mathfrak{M}_{\mathcal{A}}\mathbf{Y})\{\underline{\mathbf{A}}_n\} \cong [\mathbf{Y}, \underline{\mathbf{A}}_n]$ , that is the  $\mathcal{A}$ -cohomology of  $\mathbf{Y}$ , if we can construct a right adjoint  $\mathcal{R}_{\mathcal{A}} : \text{Map}_{\mathcal{A}, \text{d}}^{\text{op}} \rightarrow \mathcal{T}_*$  to  $\mathcal{L}_{\mathcal{A}}$  taking value in  $\mathcal{G}$ -injectives, for  $\mathcal{G} = \text{Obj } \Theta_{\mathcal{A}}$ , we can use it to produce a cosimplicial weak  $\mathcal{G}$ -resolution of any space  $\mathbf{Y}$  (cf. [Bo1, Definition 6.1]), and thus its  $\mathcal{G}$ -completion  $\hat{L}_{\mathcal{G}}\mathbf{Y}$ .

Since  $\mathcal{L}_{\mathcal{A}}$  is contravariant, we need a natural isomorphism

$$(3.11) \quad \text{Hom}_{\mathcal{C}}(\mathbf{Y}, \mathcal{R}_{\mathcal{A}}\mathfrak{X}) \cong \text{Hom}_{\text{Map}_{\mathcal{A},d}^{\text{op}}}(\mathcal{L}_{\mathcal{A}}\mathbf{Y}, \mathfrak{X}) \cong \text{Hom}_{\mathbf{Set}_*^{\Gamma}}(\mathfrak{X}, \mathcal{L}_{\mathcal{A}}\mathbf{Y})$$

for any  $\mathbf{Y} \in \mathcal{C}$  and discrete  $\mathcal{A}$ -mapping algebra  $\mathfrak{X}$ , where  $\mathbf{Set}_*^{\Gamma} = \text{Map}_{\mathcal{A},d}$ .

By (3.8), the right hand side naturally splits as a product over  $n \in \mathbb{Z}$  of pullback squares in  $\mathbf{Set}_*$ :

$$(3.12) \quad \begin{array}{ccc} M_n & \xrightarrow{\quad \boxed{\text{PB}} \quad} & \text{Hom}_{\mathbf{Set}_*}(\mathfrak{X}\{P_n\}, \mathcal{L}_{\mathcal{A}}\mathbf{Y}\{P_n\}) \\ \downarrow & & \downarrow \mathfrak{X}\{q_n\}_* \\ \text{Hom}_{\mathbf{Set}_*}(\mathfrak{X}\{A_n\}, \mathcal{L}_{\mathcal{A}}\mathbf{Y}\{A_n\}) & \xrightarrow{\mathfrak{X}\{q_n\}^*} & \text{Hom}_{\mathbf{Set}_*}(\mathfrak{X}\{P_n\}, \mathcal{L}_{\mathcal{A}}\mathbf{Y}\{A_n\}) . \end{array}$$

Since each pointed set  $\mathfrak{X}\{P_n\}$  is a coproduct in  $\mathbf{Set}_*$  of its non-zero element singletons, we can re-write this as

$$(3.13) \quad \begin{array}{ccc} M_n & \xrightarrow{\quad \boxed{\text{PB}} \quad} & \prod_{\mathfrak{X}\{P_n\}} \text{Hom}_{\mathcal{C}}(\mathbf{Y}, P\mathbf{A}_n) \\ \downarrow & & \downarrow (p\mathbf{A}_n)_{\#} \\ \prod_{\mathfrak{X}\{A_n\}} \text{Hom}_{\mathcal{C}}(\mathbf{Y}, \mathbf{A}_n) & \xrightarrow{\mathfrak{X}\{q_n\}^*} & \prod_{\mathfrak{X}\{P_n\}} \text{Hom}_{\mathcal{C}}(\mathbf{Y}, \mathbf{A}_n) . \end{array}$$

using the convention that the factor indexed by the basepoint of  $\mathfrak{X}\{P_n\}$  or  $\mathfrak{X}\{A_n\}$  is identified to zero.

Therefore, the right-hand side of (3.11) splits naturally as a product over  $n \in \mathbb{Z}$  of certain pointed sets, each of which factors in turn as a product of two types of pointed sets: namely, the products sets

$$(3.14) \quad \prod_{\mathfrak{X}\{A_n\} \setminus \text{Im } \mathfrak{X}\{q_n\}} \text{Hom}_{\mathcal{C}}(\mathbf{Y}, \mathbf{A}_n) \times \prod_{*\neq \Phi \in \mathfrak{X}\{q_n\}^{-1}(*)} \text{Hom}_{\mathcal{C}}(\mathbf{Y}, \Omega\mathbf{A}_n)$$

and pullback squares of the form:

$$(3.15) \quad \begin{array}{ccc} M_{\phi} & \xrightarrow{\quad} & \prod_{\mathfrak{X}\{q_n\}^{-1}(\phi)} \text{Hom}_{\mathcal{C}}(\mathbf{Y}, P\mathbf{A}_n) \\ \downarrow & & \downarrow (p\mathbf{A}_n)_{\#} \\ \text{Hom}_{\mathcal{C}}(\mathbf{Y}, \mathbf{A}_n) & \xrightarrow{\quad} & \prod_{\mathfrak{X}\{q_n\}^{-1}(\phi)} \text{Hom}_{\mathcal{C}}(\mathbf{Y}, \mathbf{A}_n) , \end{array}$$

for each  $* \neq \phi \in \text{Im } \mathfrak{X}\{q_n\} \subseteq \mathfrak{X}\mathcal{A}_n$ .

Thus we have a natural identification of the left-hand term  $\text{Hom}_{\mathcal{C}}(\mathbf{Y}, \mathcal{R}_{\mathcal{A}}\mathfrak{X})$  in (3.11) with a limit of sets of the form  $\text{Hom}_{\mathcal{C}}(\mathbf{Y}, -)$ , and since  $\text{Hom}_{\mathcal{C}}(\mathbf{Y}, -)$ , commutes with limits, we set

$$(3.16) \quad \mathcal{R}_{\mathcal{A}}\mathfrak{X} := \prod_{n \in \mathbb{Z}} \left( \prod_{\mathfrak{X}\{A_n\} \setminus \text{Im } \mathfrak{X}\{q_n\}} \mathbf{A}_n \times \prod_{*\neq \Phi \in \mathfrak{X}\{q_n\}^{-1}(*)} \Omega\mathbf{A}_n \times \prod_{\phi \in \text{Im } \mathfrak{X}\{q_n\} \setminus \{*\}} Q_{\phi} \right) ,$$

where for each  $* \neq \phi \in \text{Im } \mathfrak{X}\{q_n\}$  we define  $Q_\phi$  by the pullback square

$$(3.17) \quad \begin{array}{ccc} Q_\phi & \longrightarrow & \prod_{\mathfrak{X}\{q_n\}^{-1}(\phi)} P\mathbf{A}_n \\ \downarrow & & \downarrow (p_{\mathbf{A}_n})^\# \\ \mathbf{A}_n & \xrightarrow{\text{diag}} & \prod_{\mathfrak{X}\{q_n\}^{-1}(\phi)} \mathbf{A}_n \end{array}$$

in  $\mathcal{C}$ .

**3.18. Definition.** We call  $\mathcal{R}_\mathcal{A}\mathfrak{Y}$  of (3.16) the *dual Stover construction* for  $\mathcal{A}$ , applied to the discrete  $\mathcal{A}$ -mapping algebra  $\mathfrak{Y}$ . This defines a functor  $\mathcal{R}_\mathcal{A} : \text{Map}_{\mathcal{A},d}^{\text{op}} \rightarrow \mathcal{C}$ , right adjoint to  $\mathcal{L}_\mathcal{A}$ , and thus a monad  $\mathcal{T}_\mathcal{A} := \mathcal{R}_\mathcal{A} \circ \mathcal{L}_\mathcal{A}$  on  $\mathcal{C}$ , with unit  $\eta = \widehat{\text{Id}_{\mathcal{L}_\mathcal{A}}} : \text{Id} \rightarrow \mathcal{T}_\mathcal{A}$  and multiplication  $\mu = \mathcal{R}_\mathcal{A} \widehat{\text{Id}_{\mathcal{T}_\mathcal{A}}} : \mathcal{T}_\mathcal{A} \circ \mathcal{T}_\mathcal{A} \rightarrow \mathcal{T}_\mathcal{A}$ , as well as a comonad  $\mathcal{S}_\mathcal{A} := \mathcal{L}_\mathcal{A} \circ \mathcal{R}_\mathcal{A}$  on  $\text{Map}_{\mathcal{A},d}^{\text{op}}$ , with counit  $\varepsilon = \widehat{\text{Id}_{\mathcal{R}_\mathcal{A}}} : \mathcal{S}_\mathcal{A} \rightarrow \text{Id}$  and comultiplication  $\delta = \mathcal{L}_\mathcal{A} \widehat{\text{Id}_{\mathcal{S}_\mathcal{A}}} : \mathcal{S}_\mathcal{A} \rightarrow \mathcal{S}_\mathcal{A} \circ \mathcal{S}_\mathcal{A}$  (cf. [Mc, VI, §1]).

Recall that a *coalgebra* over  $\mathcal{S}_\mathcal{A}$  is an object  $\mathfrak{Y} \in \text{Map}_{\mathcal{A},d}^{\text{op}}$  equipped with a section  $\zeta : \mathfrak{Y} \rightarrow \mathcal{S}_\mathcal{A}\mathfrak{Y}$  for the counit  $\varepsilon_\mathfrak{Y} : \mathcal{S}_\mathcal{A}\mathfrak{Y} \rightarrow \mathfrak{Y}$ , such that

$$(3.19) \quad \mathcal{S}_\mathcal{A}\zeta \circ \zeta = \delta_\mathfrak{Y} \circ \zeta$$

(see [Mc, VI, §2]).

**3.20. Example.** Any *realizable* discrete  $\mathcal{A}$ -mapping algebra  $\mathcal{L}_\mathcal{A}\mathbf{Y} := \rho\mathfrak{M}_\mathcal{A}\mathbf{Y}$  has a canonical structure of a coalgebra over  $\mathcal{S}_\mathcal{A}$ , with  $\zeta : \mathcal{L}_\mathcal{A}\mathbf{Y} \rightarrow \mathcal{L}_\mathcal{A}\mathcal{R}_\mathcal{A}\mathcal{L}_\mathcal{A}\mathbf{Y}$  (in  $\text{Map}_{\mathcal{A},d}^{\text{op}}$ ) equal to  $\mathcal{L}_\mathcal{A}(\widehat{\text{Id}_{\mathcal{L}_\mathcal{A}\mathbf{Y}}})$ , where  $\widehat{\text{Id}_{\mathcal{L}_\mathcal{A}\mathbf{Y}}} : \mathbf{Y} \rightarrow \mathcal{R}_\mathcal{A}\mathcal{L}_\mathcal{A}\mathbf{Y}$  is the adjoint of  $\text{Id} : \mathcal{L}_\mathcal{A}\mathbf{Y} \rightarrow \mathcal{L}_\mathcal{A}\mathbf{Y}$ .

**3.21. The Stover category.** If  $\mathcal{A} = (\mathbf{A}_n)_{n \in \mathbb{Z}}$  is an  $\Omega$ -spectrum in  $\mathcal{C}$ , and  $\lambda$  is some infinite cardinal, we define an  $\mathcal{A}$ -*Stover object* (for  $\lambda$ ) to be a product of at most  $\lambda$  objects which are either from  $\mathcal{A}$ , or are pullbacks  $Q$  of the form:

$$(3.22) \quad \begin{array}{ccc} Q & \longrightarrow & \prod_T P\mathbf{A}_n \\ \downarrow & & \downarrow p \\ \mathbf{A}_n & \xrightarrow{\text{diag}} & \prod_T \mathbf{A}_n \end{array}$$

where  $T$  is some set of cardinality  $\leq \lambda$ .

Thus if  $\lambda$  bounds the cardinality of all sets  $\mathfrak{X}\{\mathbf{A}_n\}$  ( $n \in \mathbb{Z}$ ), then  $\mathcal{R}_\mathcal{A}\mathfrak{X}$  is an  $\mathcal{A}$ -Stover object for  $\lambda$ . Moreover, since each pullback  $Q$  in (3.22) is weakly equivalent to a product of copies of  $\Omega\mathbf{A}_n$ , we see that any  $\mathcal{A}$ -Stover object is  $\mathcal{G}$ -injective, for  $\mathcal{G} = \text{Obj } \Theta_\mathcal{A}$ .

We denote by  $\Theta_\mathcal{A}^{\text{St},\lambda}$  the full simplicial subcategory of  $\mathcal{C}$  consisting of all  $\mathcal{A}$ -Stover objects for  $\lambda$  (which include in particular all objects of  $\Theta_\mathcal{A}$ ). Note that  $\Theta_\mathcal{A}^{\text{St},\lambda}$  is itself an enriched sketch, whose mapping algebras will be called  $\mathcal{A}$ -*Stover mapping algebras* (for  $\lambda$ ). The category of  $\mathcal{A}$ -Stover mapping algebras will be denoted by  $\text{Map}_\mathcal{A}^{\text{St}}$ , and the corresponding free  $\mathcal{A}$ -Stover mapping algebra functor will be written  $\mathfrak{M}_\mathcal{A}^{\text{St}} : \mathcal{C} \rightarrow \text{Map}_\mathcal{A}^{\text{St}}$ .

Since  $\Theta_{\mathcal{A}} \subseteq \Theta_{\mathcal{A}}^{\text{St}, \lambda}$ , we have a forgetful functor  $U^{\text{St}} : \text{Map}_{\mathcal{A}}^{\text{St}} \rightarrow \text{Map}_{\mathcal{A}}$ , and when there is no danger of confusion we shall denote the composite  $\rho \circ U^{\text{St}}$  simply by  $\rho$ , so we have  $\rho \mathfrak{M}_{\mathcal{A}} = \rho \mathfrak{M}_{\mathcal{A}}^{\text{St}}$ .

**3.23. Lemma.** *The coalgebra structure map  $\zeta : \mathcal{L}_{\mathcal{A}} \mathbf{Y} \rightarrow \mathcal{L}_{\mathcal{A}} \mathcal{R}_{\mathcal{A}} \mathcal{L}_{\mathcal{A}} \mathbf{Y}$  of a realizable discrete  $\mathcal{A}$ -mapping algebra  $\mathcal{L}_{\mathcal{A}} \mathbf{Y} := \rho \mathfrak{M}_{\mathcal{A}} \mathbf{Y}$  (in  $\text{Map}_{\mathcal{A}, d}^{\text{op}}$ ), is induced by a map of  $\mathcal{A}$ -Stover mapping algebras  $\zeta' : \mathfrak{M}_{\mathcal{A}}^{\text{St}}(\mathcal{R}_{\mathcal{A}}(\rho \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathbf{Y})) \rightarrow \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathbf{Y}$  (in  $\text{Map}_{\mathcal{A}}^{\text{St}}$ ), so  $\zeta = (\rho \zeta')^{\text{op}}$ .*

*Proof.* Since  $\mathcal{R}_{\mathcal{A}}(\rho \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathbf{Y})$  is a Stover object, we see that  $\mathfrak{M}_{\mathcal{A}}^{\text{St}}(\mathcal{R}_{\mathcal{A}}(\rho \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathbf{Y}))$  is a free  $\mathcal{A}$ -Stover mapping algebra, so in order to define the map  $\zeta'$ , by Lemma 2.13 it suffices to produce a “tautological element”  $\langle \zeta' \rangle$  in

$$\begin{aligned} \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathbf{Y} \{ \mathcal{R}_{\mathcal{A}}(\rho \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathbf{Y}) \}_0 &= \text{map}_{\mathcal{C}}(\mathbf{Y}, \mathcal{R}_{\mathcal{A}}(\rho \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathbf{Y}))_0 = \text{Hom}_{\mathcal{C}}(\mathbf{Y}, \mathcal{R}_{\mathcal{A}}(\rho \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathbf{Y})) \\ &\cong \text{Hom}_{\text{Map}_{\mathcal{A}, d}^{\text{op}}}(\mathcal{L}_{\mathcal{A}} \mathbf{Y}, \mathcal{L}_{\mathcal{A}} \mathbf{Y}), \end{aligned}$$

where the last isomorphism is just (3.11). We may thus choose  $\langle \zeta' \rangle$  to be the adjoint of  $\text{Id}_{\mathcal{L}_{\mathcal{A}} \mathbf{Y}}$ .  $\square$

**3.24. Remark.** It is difficult to keep track of the (co)monads obtained from adjoint functors when they are not covariant, which is why Definition 3.18 was given in terms of  $\text{Map}_{\mathcal{A}, d}^{\text{op}}$ . However, for the purposes of the following Proposition, we prefer to work in  $\text{Map}_{\mathcal{A}, d}$  itself, so that we will refer to  $\varepsilon_{\mathfrak{Y}}^{\text{op}} : \mathfrak{Y} \rightarrow \mathcal{S}_{\mathcal{A}} \mathfrak{Y}$  rather than the original  $\varepsilon_{\mathfrak{Y}} : \mathcal{S}_{\mathcal{A}} \mathfrak{Y} \rightarrow \mathfrak{Y}$ , and so on.

**3.25. Proposition.** *For every  $\mathcal{A}$ -Stover mapping algebra  $\mathfrak{X}$ , the corresponding discrete  $\mathcal{A}$ -mapping algebra  $\mathfrak{Y} := \rho \mathfrak{X}$  has a natural structure of a coalgebra over  $\mathcal{S}_{\mathcal{A}}$ .*

Compare [BBD, Proposition 3.7] and [BB2, Proposition 3.16].

*Proof.* Using Remark 3.24, we must produce a section  $\zeta^{\text{op}} : \mathcal{S}_{\mathcal{A}} \mathfrak{Y} \rightarrow \mathfrak{Y}$  for  $\varepsilon_{\mathfrak{Y}}^{\text{op}} : \mathfrak{Y} \rightarrow \mathcal{S}_{\mathcal{A}} \mathfrak{Y}$  satisfying (3.19). In fact, we will construct a map of  $\mathcal{A}$ -Stover mapping algebras  $\zeta' : \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathcal{R}_{\mathcal{A}} \mathfrak{X} \rightarrow \mathfrak{X}$  fitting into a commuting square

$$(3.26) \quad \begin{array}{ccc} \mathfrak{M}_{\mathcal{A}}^{\text{St}}(\mathcal{R}_{\mathcal{A}}(\rho \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathcal{R}_{\mathcal{A}} \mathfrak{Y})) & \xrightarrow{\mathfrak{M}_{\mathcal{A}}^{\text{St}} \zeta'} & \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathcal{R}_{\mathcal{A}} \mathfrak{Y} \\ \delta' \downarrow & & \downarrow \zeta' \\ \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathcal{R}_{\mathcal{A}} \mathfrak{Y} & \xrightarrow{\zeta'} & \rho \mathfrak{Y} . \end{array}$$

in  $\text{Map}_{\mathcal{A}}$ , with  $\zeta = (\rho \zeta')^{\text{op}}$  and  $\delta_{\mathfrak{Y}} = \rho \delta'$ .

As in the proof of Lemma 3.23, in order to define the map

$$\delta' : \mathfrak{M}_{\mathcal{A}}^{\text{St}}(\mathcal{R}_{\mathcal{A}}(\rho \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathcal{R}_{\mathcal{A}} \mathfrak{Y})) \rightarrow \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathcal{R}_{\mathcal{A}} \mathfrak{Y},$$

by Lemma 2.13 it suffices to produce a “tautological element”  $\langle \delta' \rangle$  in

$$\mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathcal{R}_{\mathcal{A}} \mathfrak{Y} \{ \mathcal{R}_{\mathcal{A}}(\rho \mathfrak{M}_{\mathcal{A}}^{\text{St}}(\mathcal{R}_{\mathcal{A}} \mathfrak{Y})) \}_0 = \text{map}_{\mathcal{C}}(\mathcal{R}_{\mathcal{A}} \mathfrak{Y}, \mathcal{R}_{\mathcal{A}}(\rho \mathfrak{M}_{\mathcal{A}}^{\text{St}}(\mathcal{R}_{\mathcal{A}} \mathfrak{Y})))_0,$$

that is, a map  $\theta : \mathcal{R}_{\mathcal{A}} \mathfrak{Y} \rightarrow \mathcal{R}_{\mathcal{A}}(\rho \mathfrak{M}_{\mathcal{A}}^{\text{St}}(\mathcal{R}_{\mathcal{A}} \mathfrak{Y}))$ , which we choose to be the adjoint of  $\text{Id} : \rho \mathfrak{M}_{\mathcal{A}}^{\text{St}}(\mathcal{R}_{\mathcal{A}} \mathfrak{Y}) \rightarrow \rho \mathfrak{M}_{\mathcal{A}}^{\text{St}}(\mathcal{R}_{\mathcal{A}} \mathfrak{Y})$ .

Similarly, in order to define the map  $\zeta'$  it suffices to produce a “tautological element”  $\langle \zeta \rangle$  in  $\mathfrak{X} \{ \mathcal{R}_{\mathcal{A}} \mathfrak{X} \}_0$ . Since  $\mathcal{R}_{\mathcal{A}} \mathfrak{X}$  is defined by a limit,  $\langle \zeta \rangle$  will be determined by choosing compatible elements in each component of (3.16). However, each component  $\underline{A}_n$  is indexed by an element  $\phi$  in  $\mathfrak{Y} \{ A_n \} = \mathfrak{X} \{ \underline{A}_n \}_0$ , and each

component  $P\mathbf{A}_n$  is indexed by an element  $\Phi$  in  $\mathfrak{Y}\{PA_n\} = \mathfrak{X}\{P\mathbf{A}_n\}_0$ , and all these choice are compatible, yielding the required  $\langle \zeta' \rangle$ .

The verification that (3.26) commutes is as in [BB2, Proposition 3.16].  $\square$

**3.27. Theorem.** *Assume given an  $\Omega$ -spectrum  $\mathcal{A} = (\mathbf{A}_n)_{n \in \mathbb{Z}}$  in a model category  $\mathcal{C}$  (as in §0.6), and an  $\Theta_{\mathcal{A}}$ -mapping algebra  $\mathfrak{X}$  which extends to a  $\mathcal{A}$ -Stover mapping algebra for some cardinal  $\lambda$  bounding the cardinalities of  $\mathfrak{X}\{\mathbf{A}_n\}$  ( $n \in \mathbb{Z}$ ). There then is a cosimplicial object  $\mathbf{W}^\bullet \in c\mathcal{C}$ , with each  $\mathbf{W}^n$  a Stover object for  $\lambda$  (so in particular  $\mathcal{G}$ -injective), such that the simplicial  $\Theta_{\mathcal{A}}$ -mapping algebra  $\mathfrak{M}_{\mathcal{A}}\mathbf{W}^\bullet$  is weakly equivalent to  $c(\mathfrak{X})_\bullet$ .*

Compare [BB1, Proposition 9.23] and [BB2, Theorem 3.18].

*Proof.* In our case, we may define an augmented simplicial object  $\tilde{\mathfrak{Y}}_\bullet \rightarrow \mathfrak{Y}$  in  $\text{Map}_{\mathcal{A},d}^{\text{op}}$  by iterating the corresponding comonad  $\mathcal{S}_{\mathcal{A}} : \text{Map}_{\mathcal{A},d}^{\text{op}} \rightarrow \text{Map}_{\mathcal{A},d}^{\text{op}}$  on  $\mathfrak{Y} := \rho\mathfrak{X}$ , so  $\tilde{\mathfrak{Y}}_k = \mathcal{S}_{\mathcal{A}}^{k+1}\mathfrak{Y}$  (cf. [Mc, VI, §6]). By Proposition 3.25,  $\mathfrak{Y}$  is a coalgebra over  $\mathcal{S}_{\mathcal{A}}$ , so it is  $\mathcal{S}_{\mathcal{A}}$ -projective (cf. [We, §8.6.6]), and the structure map  $\zeta_{\mathfrak{Y}} : \mathfrak{Y} \rightarrow \tilde{\mathfrak{Y}}_0 = \mathcal{S}_{\mathcal{A}}\mathfrak{Y}$  provides an extra degeneracy map  $s_{k+1} := \mathcal{S}_{\mathcal{A}}^{k+1}\zeta_{\mathfrak{Y}} : \tilde{\mathfrak{Y}}_k \rightarrow \tilde{\mathfrak{Y}}_{k+1}$  (see [We, Proposition 8.6.8]).

An explicit description of  $\tilde{\mathfrak{Y}}_\bullet \rightarrow \mathfrak{Y}$  in low dimensions is given by the following diagram in  $\text{Map}_{\mathcal{A},d}^{\text{op}}$ :

$$(3.28) \quad \begin{array}{ccccc} & & & \xrightarrow{s_0 = \delta_{\mathfrak{Y}} = \mathcal{L}_{\mathcal{A}} \widehat{\text{Id}}_{\mathcal{S}_{\mathcal{A}}\mathfrak{Y}}} & \\ & & & \searrow & \\ \tilde{\mathfrak{Y}}_{-1} := \mathfrak{Y} & \xrightarrow{s_0 = \zeta_{\mathfrak{Y}}} & \tilde{\mathfrak{Y}}_0 := \mathcal{S}_{\mathcal{A}}\mathfrak{Y} & \xrightarrow{s_1 = \mathcal{S}_{\mathcal{A}}\zeta_{\mathfrak{Y}}} & \tilde{\mathfrak{Y}}_1 := \mathcal{S}_{\mathcal{A}}^2\mathfrak{Y} \\ & \xleftarrow{d_0 = \varepsilon_{\mathfrak{Y}} = \widehat{\text{Id}}_{\mathcal{R}_{\mathcal{A}}\mathfrak{Y}}} & & \xleftarrow{d_0 = \varepsilon_{\mathcal{S}_{\mathcal{A}}\mathfrak{Y}}} & \\ & & & \nwarrow & \\ & & & \xleftarrow{d_1 = \mathcal{S}_{\mathcal{A}}\varepsilon_{\mathfrak{Y}}} & \end{array}$$

Applying the functor  $\mathcal{R}_{\mathcal{A}}$  dimensionwise to  $\tilde{\mathfrak{Y}}_\bullet \rightarrow \mathfrak{Y}$  yields an augmented simplicial object  $\mathcal{R}_{\mathcal{A}}\tilde{\mathfrak{Y}}_\bullet \rightarrow \mathcal{R}_{\mathcal{A}}\mathfrak{Y}$ . We set

$$\mathbf{W}^n := \mathcal{R}_{\mathcal{A}}\tilde{\mathfrak{Y}}_{n-1} = \mathcal{R}_{\mathcal{A}}\mathcal{S}_{\mathcal{A}}^n\mathfrak{Y} \quad \text{with } \mathbf{W}^0 := \mathcal{R}_{\mathcal{A}}\mathfrak{Y}.$$

Moreover, we have an extra map  $d^{n+1} : \mathbf{W}^n = \mathcal{R}_{\mathcal{A}}\mathcal{S}_{\mathcal{A}}^n\mathfrak{Y} \rightarrow \mathcal{R}_{\mathcal{A}}\mathcal{S}_{\mathcal{A}}^{n+1}\mathfrak{Y} = \mathbf{W}^{n+1}$ , adjoint to  $\text{Id}_{\mathcal{S}_{\mathcal{A}}^{n+1}\mathfrak{Y}}$ , so we actually obtain a cosimplicial object  $\mathbf{W}^\bullet$ .

Once again, we have an explicit description of  $\mathbf{W}^\bullet$  in low dimensions as a diagram in  $\mathcal{C}$ :

$$(3.29) \quad \begin{array}{ccccc} & & & \xrightarrow{d^0 = \mathcal{R}_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}\widehat{\text{Id}}_{\mathcal{S}_{\mathcal{A}}^2\mathfrak{Y}}} & \\ & & & \searrow & \\ \mathbf{W}^0 = \mathcal{R}_{\mathcal{A}}\mathfrak{Y} & \xrightarrow{d^1 = \widehat{\text{Id}}_{\mathcal{S}_{\mathcal{A}}\mathfrak{Y}}} & \mathbf{W}^1 = \mathcal{R}_{\mathcal{A}}\mathcal{S}_{\mathcal{A}}\mathfrak{Y} & \xrightarrow{d^2 = \widehat{\text{Id}}_{\mathcal{S}_{\mathcal{A}}\mathcal{S}_{\mathcal{A}}\mathfrak{Y}}} & \mathbf{W}^2 = \mathcal{R}_{\mathcal{A}}\mathcal{S}_{\mathcal{A}}^2\mathfrak{Y} \\ & \xleftarrow{s^0 = \mathcal{R}_{\mathcal{A}}\varepsilon_{\mathfrak{Y}}} & & \xleftarrow{s^0 = \mathcal{R}_{\mathcal{A}}\varepsilon_{\mathcal{S}_{\mathcal{A}}\mathfrak{Y}}} & \\ & & & \nwarrow & \\ & & & \xleftarrow{s^1 = \mathcal{R}_{\mathcal{A}}\mathcal{S}_{\mathcal{A}}\varepsilon_{\mathfrak{Y}}} & \end{array}$$

As we saw in the proof of Proposition 3.25, the opposite of  $\mathcal{S}_{\mathcal{A}}$ -coalgebra structure map of  $\mathfrak{Y} = \rho\mathfrak{X}$  lifts to a map of  $\mathcal{A}$ -Stover mapping algebras  $\zeta' : \mathfrak{M}_{\mathcal{A}}^{\text{St}}\mathbf{W}^0 \rightarrow \mathfrak{X}$ , making  $\mathfrak{M}_{\mathcal{A}}\mathbf{W}^\bullet \rightarrow \mathfrak{X}$  into an augmented simplicial  $\mathcal{A}$ -Stover mapping algebra, which is described in low dimensions by the diagram (in  $\text{Map}_{\mathcal{A}}^{\text{St}}$ ):

$$(3.30) \quad \mathfrak{X} \xleftarrow{\zeta'} \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathbf{W}^0 \begin{array}{c} \xrightarrow{s_0 = \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathcal{R}_{\mathcal{A}} \varepsilon_{\mathfrak{Y}}} \\ \xleftarrow{d_0 = \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathcal{R}_{\mathcal{A}} \zeta_{\mathfrak{Y}}} \\ \xrightarrow{d_1 = \mathfrak{M}_{\mathcal{A}}^{\text{St}} \text{Id}_{S_{\mathcal{A}} \mathfrak{Y}}} \end{array} \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathbf{W}^1 \dots$$

By construction, applying  $\rho$  to this yields  $\tilde{\mathfrak{V}}_{\bullet} \rightarrow \mathfrak{Y}$  (viewed as a coaugmented cosimplicial object  $\text{Map}_{\mathcal{A},d}^{\text{op}}$ ). Since each  $\underline{\mathbf{A}}_n$  of  $\mathcal{A}$  is an (infinite) loop space, for each  $n, k \geq 0$ , each of the pointed sets  $\tilde{\mathfrak{V}}_k\{A_n\}$  (in the notation of §3.7) is equipped with a binary operation  $\mu_{\#} : \tilde{\mathfrak{V}}_k\{A_n\} \times \tilde{\mathfrak{V}}_k\{A_n\} \rightarrow \tilde{\mathfrak{V}}_k\{A_n\}$  (induced by the  $H$ -space multiplication  $\mu : \underline{\mathbf{A}}_n \times \underline{\mathbf{A}}_n \rightarrow \underline{\mathbf{A}}_n$ ), and we can therefore use Lemma 3.4 to calculate the homotopy groups of the augmented simplicial (abelian) group

$$G_{\bullet} := \pi_0(\mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathbf{W}^{\bullet}\{\underline{\mathbf{A}}_n\}) \xrightarrow{\zeta'\{\underline{\mathbf{A}}_n\}_{\#}} \pi_0 \mathfrak{X}\{\underline{\mathbf{A}}_n\} =: G_{-1}$$

from the object  $\tilde{\mathfrak{V}}_{\bullet}\{P_n \rightarrow A_n\}$  in  $s\text{Set}_*^J$ .

Note that the indexing of (3.30) does not match with that of (3.28), even after taking into account the fact that these diagrams are in opposite categories. It will be convenient to choose our indexing convention for the augmented simplicial object  $\rho \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathbf{W}^{\bullet} \rightarrow \mathfrak{Y}$  (in  $s\text{Map}_{\mathcal{A}}^{\text{St}}$ ) so that the extra degeneracy map is indexed as:

$$s_{-1}^k = (\varepsilon_{S_{\mathcal{A}}^{k+1} \mathfrak{Y}})^{\text{op}} : \rho \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathbf{W}^k \rightarrow \rho \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathbf{W}^{k+1}$$

with

$$(3.31) \quad d_i s_{-1} = \begin{cases} \text{Id} & \text{if } i = 0 \\ s_{-1} d_{i-1} & \text{if } i \geq 1 \end{cases}$$

(which follows from the fact that  $\varepsilon_{\mathfrak{Y}} \circ \zeta_{\mathfrak{Y}} = \text{Id}$  in  $\text{Map}_{\mathcal{A},d}^{\text{op}}$ , and  $\varepsilon$  is a natural transformation).

If we knew that  $s_{-1}^k$  induced a group homomorphism  $G_k \rightarrow G_{k+1}$  for each  $k \geq -1$ , we could deduce directly that the augmented simplicial group  $G_{\bullet} \rightarrow G_{-1}$  is acyclic. However, unlike the rest of the simplicial structure on  $\tilde{\mathfrak{V}}_{\bullet}$ , the maps  $s_{-1}$  are not induced from maps of full  $\mathcal{A}$ -Stover mapping algebras, so they need not respect the  $H$ -space structure induced by that of  $\underline{\mathbf{A}}_n$ .

Nevertheless, we can modify the usual proof of the acyclicity of  $G_{\bullet} \rightarrow G_{-1}$  as follows: any Moore cycle  $\alpha \in C_k G_{\bullet} \subseteq G_k$  (with  $d_i \alpha = 0$  for  $0 \leq i \leq k$ ) may be represented by an element  $a \in \tilde{\mathfrak{V}}_k\{A_n\}$ , equipped with elements  $b_i \in \tilde{\mathfrak{V}}_{k-1}\{P_n\}$  such that  $d_i a = (\tilde{\mathfrak{V}}_{k-1}\{q_n\})(b_i)$  ( $0 \leq i \leq k$ ).

If we let  $c := s_{-1}^k a \in \tilde{\mathfrak{V}}_{k+1}\{A_n\}$ , by (3.31) we have  $d_0(c) = a$  and

$$d_i c = d_i s_{-1}^k a = s_{-1}^{k-1} d_i a = s_{-1}^{k-1} (\tilde{\mathfrak{V}}_{k-1}\{q_n\})(b_i) = (\tilde{\mathfrak{V}}_{k-1}\{q_n\})(s_{-1}^{k-1} b_i) \text{ for } 1 \leq i,$$

so that  $[c] \in \pi_0 \tilde{\mathfrak{V}}_{k+1}\{A_n\} = G_{k+1}$  is a Moore chain, with  $\partial_{k+1}([c]) = \alpha$ .

Thus we have shown that  $\zeta' : \mathfrak{M}_{\mathcal{A}}^{\text{St}} \mathbf{W}^{\bullet} \rightarrow c(\mathfrak{X})_{\bullet}$  is a weak equivalence of simplicial  $\mathcal{A}$ -Stover mapping algebras.  $\square$

**3.32. Definition.** The *total space*  $\text{Tot } \mathbf{W}^{\bullet}$  of a cosimplicial space  $\mathbf{W}^{\bullet} \in c\mathcal{T}$  is defined to be the simplicial mapping space  $\text{map}_{\mathcal{T}}(\Delta^{\bullet}, \mathbf{W}^{\bullet})$ , where  $\Delta^{\bullet} \in c\mathcal{T}$  has  $\Delta^n := \Delta[n] \in \mathcal{T}$  (the standard  $n$ -simplex) – see [BK1, X, §3].



**3.33. Corollary.** *If an  $\mathcal{A}$ -Stover mapping algebra  $\mathfrak{X}$  is realizable by  $\mathbf{Y} \in \mathcal{T}_*$ , its  $\mathcal{G}$ -completion  $\hat{L}_{\mathcal{G}}\mathbf{Y}$  is weakly equivalent to  $\mathrm{Tot} \mathbf{W}^\bullet$ , where  $\mathbf{W}^\bullet \in c\mathcal{T}_*$  is the cosimplicial object of Theorem 3.27.*

*Proof.* The simplicial space  $\mathbf{W}^\bullet$  is a weak  $\mathcal{G}$ -resolution of  $\mathbf{Y}$ , so in particular it is a  $\mathcal{G}$ -complete expansion (cf. [Bo1, Definition 9.4]), and thus  $\mathrm{Tot} \mathbf{W}^\bullet \simeq \hat{L}_{\mathcal{G}}\mathbf{Y}$  by [Bo1, Theorem 9.5].  $\square$

**3.34. Example.** In particular, if  $\mathbf{Y}$  is  $\mathcal{G}$ -good (cf. [Bo1, Definition 8.3]) then  $\mathfrak{X}$  is weakly equivalent to  $\mathfrak{M}_{\mathcal{A}}\hat{L}_{\mathcal{G}}\mathbf{Y}$  by [Bo1, Proposition 8.5].

This will hold, for example, when  $\mathcal{C} = \mathcal{T}_*$  and  $\mathcal{G} = \mathrm{Obj} \Theta_R$  for  $R \subseteq \mathbb{Q}$  or  $R = \mathbb{F}_p$ , if  $\mathbf{Y} \in \mathcal{T}_*$  is  $R$ -perfect (e.g., simply connected), by [BK1, VII, §3]. In this case  $\hat{L}_{\mathcal{G}}\mathbf{Y} = R_\infty\mathbf{Y}$  is just the  $R$ -completion.

**3.35. Proposition.** *Let  $\mathcal{G} = \mathrm{Obj} \Theta_{\mathcal{A}}$ , for  $\mathcal{A} = (\underline{\mathbf{A}}_n)_{n=0}^\infty$  an  $\Omega$ -spectrum model of a connective ring spectrum, with  $\pi_0\mathcal{A}$  commutative, and  $R = \mathrm{core} \pi_0\mathcal{A}$  (cf. [BK2]). If a space  $\mathbf{Y} \in \mathcal{T}_*$  is  $R$ -good, it is  $\mathcal{G}$ -good, and  $\hat{L}_{\mathcal{G}}\mathbf{Y} = R_\infty\mathbf{Y}$  is the usual  $R$ -completion of  $\mathbf{Y}$ .*

*Proof.* This follows from Corollary 3.33 and [Bo1, Theorem 9.7].  $\square$

**3.36. Remark.** Theorem 3.27 thus provides us with a *recovery* procedure for retrieving  $\mathbf{Y}$  from the  $\Theta_{\mathcal{A}}$ -mapping algebra  $\mathfrak{X} = \mathfrak{M}_{\mathcal{A}}\mathbf{Y}$ , up to  $\mathcal{G}$ -completion – that is, it shows that the  $\mathcal{A}$ -Stover mapping algebra structure on  $X = \mathrm{map}_{\mathcal{C}}(\mathbf{Y}, \mathbf{A})$  suffices to recover  $\mathbf{Y}$  (precisely to the extent that  $\mathbf{Y}$  is determined by  $X$  – namely, up to  $\mathcal{G}$ -completion) – although it does not allow us to *recognize* when an abstract  $\Theta_{\mathcal{A}}$ -mapping algebra is in fact realizable.

Moreover, it implies that we can actually define a  $\mathcal{G}$ -completion for an *abstract*  $\mathcal{A}$ -Stover mapping algebra  $\mathfrak{X}$ : namely,  $\hat{L}_{\mathcal{G}}\mathfrak{X} := \mathrm{Tot} \mathbf{W}^\bullet$  for  $\mathbf{W}^\bullet$  as in the Theorem. However, without additional assumptions, it need not be true that  $\mathfrak{X} \simeq \mathfrak{M}_{\mathcal{A}}(\hat{L}_{\mathcal{G}}\mathfrak{X})$ .

#### 4. REALIZING SIMPLICIAL $\Theta$ -ALGEBRA RESOLUTIONS

Our goal here is to show how a free simplicial (algebraic) resolution  $V_\bullet$  of an enrichable  $\Theta$ -algebra  $\Lambda$  can be realized in  $\mathcal{C}$ . For this purpose, we require the following:

**4.1. Definition.** For any  $\mathfrak{G}$ -sketch  $\Theta$ , a *CW-resolution* of a  $\Theta$ -algebra  $\Lambda \in \Theta\text{-Alg}$  is a cofibrant replacement  $\varepsilon : G_\bullet \xrightarrow{\sim} c\Lambda$  (in the model category of simplicial  $\Theta$ -algebras given by Proposition 2.5), equipped with a CW basis  $(\overline{G}_n)_{n=0}^\infty$  (as in §1.9), with each  $\overline{G}_n$  a free  $\Theta$ -algebra.

**4.2. Remark.** In fact, any CW object  $G_\bullet$  for which each  $\overline{G}_n$  is a free  $\Theta$ -algebra, and each attaching map  $\overline{\partial}_0^{G_n} (n \geq 0)$  surjects onto  $Z_{n-1}G_\bullet$ , is a CW-resolution. Here we set  $Z_{-1}G_\bullet := \Lambda$  and  $\overline{\partial}_0^{G_0} := \varepsilon$ , so that

$$(4.3) \quad \varepsilon \circ \overline{\partial}_0^{G_1} = 0.$$

**4.4. Lemma.** *Let  $\mathbf{W}^\bullet \in c\mathcal{C}$  be a Reedy cofibrant cosimplicial object over a model category  $\mathcal{C}$  (as in §0.6), and  $\mathbf{B}$  a homotopy group object in  $\mathcal{C}$ . Then for any Moore chain  $\beta \in C_n[\mathbf{W}^\bullet, \mathbf{B}]$  for the simplicial group  $[\mathbf{W}^\bullet, \mathbf{B}]$ :*

- (a)  $\beta$  can be realized by a map  $b : \mathbf{W}^n \rightarrow \mathbf{B}$  with  $b \circ d_{n-1}^i = 0$  for all  $1 \leq i \leq n$ .
- (b) If  $\beta$  is an algebraic Moore cycle, we can choose a nullhomotopy  $H : \mathbf{W}^{n-1} \rightarrow PB \subseteq \mathbf{B}^{[0,1]}$  for  $b \circ d_{n-1}^0$  such that  $H \circ d_{n-2}^j = 0$  for  $1 \leq j \leq n-1$ .

*Proof.* Since  $\mathbf{W}^\bullet$  is Reedy cofibrant, the simplicial space  $\mathbf{U}_\bullet = \text{map}_*(\mathbf{W}^\bullet, \mathbf{B}) \in s\mathcal{S}_*$  is Reedy fibrant (cf. §1.16), so by [Sto, Lemma 2.7], for every  $i \geq 0$  the inclusion  $\iota : C_n \mathbf{U}_\bullet \hookrightarrow U_n$  induces an isomorphism

$$(4.5) \quad \iota_* : \pi_i C_n \mathbf{U}_\bullet \rightarrow C_n \pi_i \mathbf{U}_\bullet$$

(cf. §1.6). Thus we can represent  $\alpha \in C_n \pi_0 \mathbf{U}_\bullet$  by a map  $a \in C_n \mathbf{U}_\bullet$ , which implies (i).

If  $\alpha$  is a cycle, then  $\partial_n(\alpha) = [a \circ d_{n-1}^0]$  vanishes in  $\pi_0 C_{n-1} \mathbf{U}_\bullet$ , so we have a nullhomotopy  $H$  for  $a \circ d_{n-1}^0$  in

$$PC_{n-1} \text{map}_*(\mathbf{W}^\bullet, \mathbf{B}) = C_{n-1} \text{map}_*(\mathbf{W}^\bullet, P\mathbf{B}) \subseteq \text{map}_*(\mathbf{W}^{n-1}, P\mathbf{B}),$$

which implies (ii).  $\square$

The following result essentially dualizes (and extends) [Bl, Theorem 3.16]:

**4.6. Theorem.** *Assume given an enriched sketch  $\Theta$  in a model category  $\mathcal{C}$  (as in §0.6), with  $\Theta := \pi_0 \Theta$  the associated “algebraic” sketch, and let  $\Lambda$  be a  $\Theta$ -algebra equipped with a CW-resolution  $V_\bullet$ . If  $\Lambda = \Lambda_{\mathfrak{X}}$  is enriched by a  $\Theta$ -mapping algebra  $\mathfrak{X}$ , then there is a CW cosimplicial object  $\mathbf{W}^\bullet \in c\mathcal{C}$  realizing  $V_\bullet$  – that is, there is an augmentation  $\varepsilon_{[\infty]} : \mathfrak{M}_\Theta \mathbf{W}^0 \rightarrow \mathfrak{X}$  such that  $\pi_0(\mathfrak{M}_\Theta \mathbf{W}^\bullet) \rightarrow \pi_0 \mathfrak{X}$  is isomorphic to  $V_\bullet \rightarrow \Lambda$ . If  $\mathfrak{X} = \mathfrak{M}_\Theta \mathbf{Y}$  for some  $\mathbf{Y} \in \mathcal{C}$ , then in  $V_\bullet \rightarrow \Lambda$  can be realized by a coaugmented cosimplicial object  $\mathbf{Y} \rightarrow \mathbf{W}^\bullet$ .*

**4.7. Remark.** The cardinal  $\lambda$  which bounds the size of the products in  $\Theta$  must be chosen so that all the free  $\Theta$ -algebras  $\overline{V}_n$  in the CW basis for  $V_\bullet$  are represented by objects in  $\Theta$  (see §2.17).

*Proof.* We first choose once and for all objects  $\overline{\mathbf{W}}^n$  in  $\Theta$  realizing  $\overline{V}_n$ , in the sense of §2.17 – so  $\pi_0 \mathfrak{M}_\Theta \overline{\mathbf{W}}^n \cong \overline{V}_n$  as  $\Theta$ -algebras. This is possible by assumption 4.7.

We will construct the cosimplicial object  $\mathbf{W}^\bullet \in c\mathcal{C}$  by a double induction: in the outer induction, we construct a sequence of cosimplicial objects and maps:

$$(4.8) \quad \dots \rightarrow \mathbf{W}_{[n]}^\bullet \xrightarrow{\pi_{[n]}} \mathbf{W}_{[n-1]}^\bullet \xrightarrow{\pi_{[n-1]}} \mathbf{W}_{[n-2]}^\bullet \rightarrow \dots \rightarrow \mathbf{W}_{[0]}^\bullet,$$

such that:

- (a)  $\mathbf{W}^\bullet$  is the dimensionwise limit of (4.8).
- (b) Each cosimplicial object  $\mathbf{W}_{[n]}^\bullet$  is weakly  $\mathcal{G}$ -fibrant for  $\mathcal{G} := \text{Obj } \Theta$  (cf. §1.18), as well as being cofibrant in the Reedy model category (§1.16).
- (c) The simplicial  $\Theta$ -mapping algebra  $\mathfrak{V}_{[n]}^\bullet := \mathfrak{M}_\Theta \mathbf{W}_{[n]}^\bullet$  has an augmentation  $\varepsilon_{[n]} : \mathfrak{V}_{[n]}^0 = \mathfrak{M}_\Theta \mathbf{W}_{[n]}^0 \rightarrow \mathfrak{X}$ , which we can identify with a 0-simplex  $\overline{\varepsilon_{[n]}}$  in  $(\mathfrak{X}\{\mathbf{W}_{[n]}^0\})_0$  by Lemma 2.13.
- (d)  $\mathbf{W}_{[n]}^\bullet$  is an  $n$ -coskeletal weak CW cosimplicial object, with CW basis  $(\overline{\mathbf{W}}^k)_{k=0}^n$  (with zero CW basis object in dimensions  $> n$ ).
- (e) The augmented simplicial  $\Theta$ -mapping algebra  $\mathfrak{M}_\Theta \mathbf{W}_{[n]}^\bullet$  realizes  $V_\bullet \rightarrow \Lambda$  through simplicial dimension  $n$ .
- (f) The maps  $\pi_{[n]}$  restrict to a fibration weak equivalence  $\pi_{[n]}^k : \mathbf{W}_{[n]}^k \rightarrow \mathbf{W}_{[n-1]}^k$  for each  $0 \leq k < n$ , so  $\mathbf{W}^k$  is the homotopy limit of the objects  $\mathbf{W}_{[n]}^k$  ( $n \geq 0$ ).

- (g) The augmentation  $\varepsilon_{[n-1]} : \mathfrak{W}_{\bullet}^{[n-1]} \rightarrow \mathfrak{X}$  extends along the  $\Theta$ -mapping algebra map  $\pi_{[n]}^* : \mathfrak{W}_{\bullet}^{[n-1]} \rightarrow \mathfrak{W}_{\bullet}^{[n]}$  to  $\varepsilon_{[n]} : \mathfrak{W}_{\bullet}^{[n]} \rightarrow \mathfrak{X}$ .

### Step 0 of the outer induction.

We start the induction with  $\mathbf{W}_{[0]}^{\bullet} := c(\overline{\mathbf{W}}^0)^{\bullet}$  (the constant cosimplicial object), which is both Reedy cofibrant and weakly  $\mathcal{G}$ -fibrant. Note that because  $\overline{V}_0$  is a free  $\Theta$ -algebra, the  $\Theta$ -algebra augmentation  $\varepsilon : \overline{V}_0 \rightarrow \Lambda$  corresponds to a unique element in  $[\varepsilon_{[0]}] \in \Lambda\{\overline{V}_0\} = \pi_0 \mathfrak{X}\{\overline{\mathbf{W}}^0\}$ , for which we may choose a representative  $\overline{\varepsilon}_{[0]} \in \mathfrak{X}\{\overline{\mathbf{W}}^0\}_0$ , which by Lemma 2.13 corresponds to a map of  $\Theta$ -mapping algebras  $\varepsilon_{[0]} : \mathfrak{W}_{[0]}^0 = \mathfrak{M}_{\Theta} \overline{\mathbf{W}}^0 \rightarrow \mathfrak{X}$ .

### Step 1 of the outer induction.

We choose a map  $\overline{d}_0^0 : \overline{\mathbf{W}}^0 \rightarrow \overline{\mathbf{W}}^1$  realizing the first attaching map  $\overline{\partial}_0^1 : \overline{V}_1 \rightarrow V_0 = \overline{V}_0$ , and define (the 1-truncation of)  $\mathbf{W}_{[1]}^{\bullet}$  by the diagram:

$$(4.9) \quad \begin{array}{ccccc} \mathbf{W}_{[1]}^0 & = & \overline{\mathbf{W}}^0 & \times & P\overline{\mathbf{W}}^1 \\ d_0^0 \left( \begin{array}{c} \uparrow \\ s^0 \\ \downarrow \end{array} \right) d_0^1 & d_0^0 = d_0^1 = \text{Id} & \downarrow & \xrightarrow{\overline{d}_0^0} & \xleftarrow{d_0^1 = p} \\ \mathbf{W}_{[1]}^1 & = & \overline{\mathbf{W}}^0 & \times & P\overline{\mathbf{W}}^1 \\ & & & \times & \downarrow d_0^0 = d_0^1 = \text{Id} \end{array}$$

Here  $p : PX \rightarrow X$  is the path fibration in  $\mathcal{C}$ , defined as in (2.10).

To define the augmentation  $\overline{\varepsilon}_{[1]}$  as a 0-simplex in  $(\mathfrak{X}\{\mathbf{W}_{[1]}^0\})_0$  extending  $\overline{\varepsilon}_{[0]} \in \mathfrak{X}\{\mathbf{W}_{[0]}^0\} = \mathfrak{X}\{\overline{\mathbf{W}}^0\}$  (see (b) above), we use the fact that

$$\mathfrak{X}\{\mathbf{W}_{[1]}^0\} = \mathfrak{X}\{\overline{\mathbf{W}}^0 \times P\overline{\mathbf{W}}^1\} = \mathfrak{X}\{\overline{\mathbf{W}}^0\} \times \mathfrak{X}\{P\overline{\mathbf{W}}^1\} = \mathfrak{X}\{\overline{\mathbf{W}}^0\} \times P\mathfrak{X}\{\overline{\mathbf{W}}^1\},$$

by §2.11(a)-(b), so we need only to find a 0-simplex  $H$  in  $P\mathfrak{X}\{\overline{\mathbf{W}}^1\}$  – which, by (2.10), is a 1-simplex in  $\mathfrak{X}\{\overline{\mathbf{W}}^1\}$  with  $d_1 H = 0$ .

In order to qualify as an augmentation  $\mathfrak{W}_{\bullet}^{[1]} \rightarrow \mathfrak{X}$  of simplicial  $\Theta$ -mapping algebras,  $\varepsilon_{[1]}$  must satisfy the simplicial identity

$$(4.10) \quad \varepsilon_{[1]} \circ d_0 = \varepsilon_{[1]} \circ d_1 : \mathfrak{W}_1^{[1]} \rightarrow \mathfrak{X}$$

as maps of  $\Theta$ -mapping algebras – or equivalently, these must correspond to the same 0-simplex in

$$\mathfrak{X}\{\mathbf{W}_{[1]}^1\} = \mathfrak{X}\{\overline{\mathbf{W}}^0 \times \overline{\mathbf{W}}^1 \times \overline{\mathbf{W}}^1\} = \mathfrak{X}\{\overline{\mathbf{W}}^0\} \times \mathfrak{X}\{\overline{\mathbf{W}}^1\} \times \mathfrak{X}\{\overline{\mathbf{W}}^1\}.$$

In the first factor and third factor this obviously holds, so we need only consider the two 0-simplices  $\mathfrak{X}\{\overline{\mathbf{W}}^1\}$ : in other words, since the path fibration  $p$  in (4.9) becomes  $d_0$  in the simplicial set  $\mathfrak{X}\{\overline{\mathbf{W}}^1\}$  (since it is induced by an inclusion  $\Delta[0] \hookrightarrow \Delta[1]$ ), we must choose the nullhomotopy  $H$  so that  $d_0 H$  is the 0-simplex  $(\overline{d}_0^0)_{\#} \overline{\varepsilon}_{[0]}$ .

But by (4.3) we know that  $\varepsilon \circ \overline{\partial}_0^0 = 0$  in  $\Theta\text{-Alg}$ , which implies (by our choices of  $\overline{d}_0^0$  and  $\overline{\varepsilon}_{[0]}$  representing  $\overline{\partial}_0^0$  and  $\varepsilon$ , respectively) that  $(\overline{d}_0^0)_{\#} \overline{\varepsilon}_{[0]}$  is nullhomotopic, so we can choose an  $H$  as required.

### Step $n$ of the outer induction ( $n \geq 2$ ):

Assume given  $\mathbf{W}_{[n-1]}^\bullet$  satisfying (a)-(g) above, we construct an intermediate  $n$ -coskeletal restricted cosimplicial object  $\mathbf{W}_{[n]}^\bullet$  (cf. §1.1) by a descending induction on the cosimplicial dimension  $0 \leq k \leq n$ .

We require that, for each  $0 \leq k < n$ , the object  $\mathbf{W}_{[n]}^k \in \mathcal{C}$  is defined by the (homotopy) pullback diagram:

$$(4.11) \quad \begin{array}{ccc} \mathbf{W}_{[n]}^k & \xrightarrow{q^k} & (\Omega^{n-k-1} \overline{\mathbf{W}}^n)^{\Delta[1]} \\ \psi_{[n]}^k \downarrow \simeq & \boxed{\text{PB}} & \downarrow \text{ev}_0 \simeq \\ \mathbf{W}_{[n-1]}^k & \xrightarrow{\eta_k} & \Omega^{n-k-1} \overline{\mathbf{W}}^n \end{array}$$

for some map  $\eta_k$  such that

$$(4.12) \quad \eta_k \circ d_{k-1}^i = 0 \quad \text{for all } 1 \leq i \leq k .$$

The idea is that the projection of the coface map  $\underline{d}_{k-1}^0 : \mathbf{W}_{[n-1]}^{k-1} \rightarrow \mathbf{W}_{[n]}^k$  onto  $\Omega^{n-k-1} \overline{\mathbf{W}}^n$  in (4.11) describes the value  $a_{k-1}$  of a certain “universal  $(n - k - 1)$ -th order cohomology operation” (see §4.32 below), while the projection onto  $(\Omega^{n-k-1} \overline{\mathbf{W}}^n)^{\Delta[1]}$  describes a homotopy  $H_{k-1}$  between this value and the corresponding element  $\eta_k \circ d_{k-1}^0$  in the cohomology of  $\mathbf{W}_{[n-1]}^{k-1}$ . This higher order operation is defined by composing the homotopy of a lower order operation with some map (which we think of as a primary cohomology operation) – in this case,  $H_k \circ d_{k-1}^0$ .

At the  $k$ -th stage, we assume that we have defined  $\mathbf{W}_{[n]}^i$  for  $n \geq i \geq k + 1$ , with all coface maps  $d_i^j : \mathbf{W}_{[n]}^i \rightarrow \mathbf{W}_{[n]}^{i+1}$  for all  $j$  and  $n \geq i > k + 1$ , as well as  $\underline{d}_k^0 : \mathbf{W}_{[n-1]}^k \rightarrow \mathbf{W}_{[n]}^{k+1}$  (if  $k < n$ ), with the 0-th coface map of  $\mathbf{W}_{[n]}^\bullet$  given by:

$$(4.13) \quad \underline{d}_k^0 := \underline{d}_k^0 \circ \psi_{[n]}^k : \mathbf{W}_{[n]}^k \rightarrow \mathbf{W}_{[n]}^{k+1} .$$

We write:

$$(4.14) \quad H_k := q^{k+1} \circ \underline{d}_k^0 : \mathbf{W}_{[n-1]}^k \rightarrow (\Omega^{n-k-2} \overline{\mathbf{W}}^n)^{\Delta[1]}$$

for the homotopy given in the previous stage, and:

$$(4.15) \quad a_k := \text{ev}_1 \circ H_k : \mathbf{W}_{[n-1]}^k \rightarrow \Omega^{n-k-2} \overline{\mathbf{W}}^n$$

for the previous value of the corresponding higher order operation, so

$$(4.16) \quad H_k : \eta_{k+1} \circ d_k^0 \sim a_k .$$

We also assume by induction that:

$$(4.17) \quad H_k \circ d_{k-1}^i = 0 \text{ for all } 1 \leq i \leq k .$$

This implies that  $a_k \circ d_{k-1}^i = 0$  for  $i \geq 1$ , but in fact we require that:

$$(4.18) \quad a_k \circ d_{k-1}^i = 0 \text{ for all } 0 \leq i \leq k .$$

**Step  $k = n$  of the descending induction:**

We start the induction by setting

$$(4.19) \quad \mathbf{W}_{[n]}^n := \mathbf{W}_{[n-1]}^n \times \overline{\mathbf{W}}^n.$$

By assumption  $\mathbf{W}_{[n-1]}^\bullet$  is Reedy cofibrant, so the bisimplicial set

$$\mathbf{U}_\bullet := \text{map}_c(\mathbf{W}_{[n-1]}^\bullet, \overline{\mathbf{W}}^n)$$

is Reedy fibrant. Moreover, since  $\pi_0 \mathfrak{M}_\Theta \mathbf{W}_{[n-1]}^k \cong V_k$  for all  $0 \leq k < n$  by (c) above, the algebraic attaching map  $\overline{\partial}_0^n : \overline{V}_n \rightarrow V_{n-1}$  is a homotopy class

$$(4.20) \quad \alpha \in \pi_0 \mathbf{U}_{n-1} = [\mathbf{W}_{[n-1]}^{n-1}, \overline{\mathbf{W}}^n] = \pi_0(\mathfrak{M}_\Theta \mathbf{W}_{[n-1]}^{n-1} \{\overline{\mathbf{W}}^n\}) = V_{n-1} \{\overline{\mathbf{W}}^n\},$$

where the last equality follows from Lemma 2.19. This  $\alpha$  is a Moore chain in  $\pi_0 \mathbf{U}_\bullet$  by Definition 1.9, so by Lemma 4.4(a),  $\overline{\partial}_0^n$  can be represented by a continuous map  $\overline{d}_{n-1}^0 : \mathbf{W}_{[n-1]}^{n-1} \rightarrow \overline{\mathbf{W}}^n$  satisfying:

$$(4.21) \quad \overline{d}_{n-1}^0 \circ d_{n-2}^j = 0 \text{ for all } 1 \leq j \leq n-1,$$

This defines  $\underline{d}_{n-1}^0 : \mathbf{W}_{[n-1]}^{n-1} \rightarrow \mathbf{W}_{[n]}^n$  into the product (4.19), extending the given face map  $d_{n-1}^0 : \mathbf{W}_{[n-1]}^{n-1} \rightarrow \mathbf{W}_{[n-1]}^n$ .

#### Step $k = n - 1$ of the descending induction:

We define  $\mathbf{W}_{[n]}^{n-1}$  by the pullback diagram (4.11), with  $\eta_{n-1} = 0$ . Thus:

$$(4.22) \quad \mathbf{W}_{[n]}^{n-1} := \mathbf{W}_{[n-1]}^{n-1} \times P\overline{\mathbf{W}}^n.$$

By (1.10), the class  $\alpha$  of (4.20) is in fact a Moore cycle, so by Lemma 4.4(b) we have a nullhomotopy

$$(4.23) \quad H_{n-2} : \overline{d}_{n-1}^0 \circ d_{n-2}^0 \sim 0$$

satisfying (4.17).

We define  $\underline{d}_{n-2}^0 : \mathbf{W}_{[n-1]}^{n-2} \rightarrow \mathbf{W}_{[n]}^{n-1}$  into the new factor  $P\overline{\mathbf{W}}^n$  (and extending the given face map  $d_{n-2}^0 : \mathbf{W}_{[n-1]}^{n-2} \rightarrow \mathbf{W}_{[n-1]}^{n-1}$ ) to be

$$(4.24) \quad H_{n-2} : \mathbf{W}_{[n-1]}^{n-2} \rightarrow P\overline{\mathbf{W}}^n.$$

We define the coface map  $\underline{d}_{n-1}^1 : \mathbf{W}_{[n]}^{n-1} \rightarrow \mathbf{W}_{[n]}^n$  by the given  $d_{n-1}^1 : \mathbf{W}_{[n-1]}^{n-1} \rightarrow \mathbf{W}_{[n-1]}^n$  into the first factor of (4.19), and the composite

$$(4.25) \quad \mathbf{W}_{[n-1]}^{n-1} \xrightarrow{\text{proj}_{P\overline{\mathbf{W}}^n}} P\overline{\mathbf{W}}^n \xrightarrow{p} \overline{\mathbf{W}}^n,$$

onto the second factor of (4.19) (where  $p : P\overline{\mathbf{W}}^n \rightarrow \overline{\mathbf{W}}^n$  is the path fibration).

The remaining face maps  $\underline{d}_{n-1}^i : \mathbf{W}_{[n]}^{n-1} \rightarrow \mathbf{W}_{[n]}^n$  ( $2 \leq i \leq n$ ) extend the given  $d_{n-1}^i : \mathbf{W}_{[n-1]}^{n-1} \rightarrow \mathbf{W}_{[n-1]}^n$  by the zero map into the CW basis  $\overline{\mathbf{W}}^n$ .

The only cosimplicial identity that can be verified at this stage is

$$\underline{d}_{n-1}^1 \underline{d}_{n-2}^0 = \underline{d}_{n-1}^0 \underline{d}_{n-2}^0,$$

which follows from the fact that  $p \circ H_{n-2} = \overline{d}_{n-1}^0 \circ d_{n-2}^0$ , by (4.23).

**Step  $k$  of the descending induction** ( $0 < k \leq n-2$ ):

We define the map  $\eta_k : \mathbf{W}_{[n-1]}^k \rightarrow \Omega^{n-k-1}\overline{\mathbf{W}}^n$  as follows:

By assumption we are given a homotopy  $H_k : \eta_{k+1} \circ d_k^0 \sim a_k$ , so we get a homotopy:

$$(4.26) \quad H_k \circ d_{k-1}^0 : \eta_{k+1} \circ d_k^0 \circ d_{k-1}^0 \sim a_k \circ d_{k-1}^0$$

where  $\eta_{k+1} \circ d_k^0 \circ d_{k-1}^0 : \mathbf{W}_{[n-1]}^{k-1} \rightarrow \overline{\mathbf{W}}_{[n-1]}^{k+1}$  is the zero map by (4.12), Definition 1.12(c), and the identity  $d^1 d^0 = d^0 d^0$ ; and  $a_k \circ d_{k-1}^0 = 0$  by (4.18). Therefore,  $H_k \circ d_{k-1}^0 : \mathbf{W}_{[n-1]}^{k-1} \rightarrow (\Omega^{n-k-2}\overline{\mathbf{W}}^n)^{\Delta[1]}$  is a self-nullhomotopy, so it factors through the inclusion  $i_k : \Omega^{n-k-1}\overline{\mathbf{W}}^n \hookrightarrow (\Omega^{n-k-2}\overline{\mathbf{W}}^n)^{\Delta[1]}$  and thus defines a map  $a_{k-1} : \mathbf{W}_{[n-1]}^{k-1} \rightarrow \Omega^{n-k-1}\overline{\mathbf{W}}^n$  with

$$(4.27) \quad i_k \circ a_{k-1} = H_k \circ d_{k-1}^0.$$

Moreover,

$$i_k \circ a_{k-1} \circ d_{k-2}^i = H_k \circ d_{k-1}^0 \circ d_{k-2}^i = H_k \circ d_{k-1}^{i+1} \circ d_{k-2}^0 = 0$$

for all  $0 \leq i \leq k-1$  by (4.17), so (4.18) holds for  $a_{k-1}$ .

Therefore,  $a_{k-1}$  is a  $(k-1)$ -cycle for the Reedy fibrant bisimplicial set  $\mathbf{U}_\bullet := \text{map}_{\mathcal{C}}(\mathbf{W}_{[n-1]}^\bullet, \Omega^{n-k-1}\overline{\mathbf{W}}^n)$ , and thus in particular represents a  $(k-1)$ -cycle  $[a_{k-1}]$  for  $V_\bullet\{\Omega^{n-k-1}\overline{\mathbf{W}}^n\}$ , as in (4.20).

Because  $V_\bullet \rightarrow \Lambda$  is a resolution, and thus acyclic, there is a class  $\gamma_k \in \overline{V}_k\{\Omega^{n-k-1}\overline{\mathbf{W}}^n\}$  with  $\overline{\partial}_0^k(\gamma_k) = [a_{k-1}]$ . Moreover, the map  $\overline{\varphi}_{[n-1]}^k : \mathbf{W}_{[n-1]}^k \rightarrow \overline{\mathbf{W}}^k$  induces the inclusion  $(\overline{\varphi}_{[n-1]}^k)^* : \overline{V}_k \hookrightarrow V_k$ , so we have a class  $[\eta_k] := (\overline{\varphi}_{[n-1]}^k)^*(\gamma_k) \in V_k\{\Omega^{n-k-1}\overline{\mathbf{W}}^n\}$  which is a Moore chain by §1.12(b).

By Lemma 4.4(a) we can represent this class by a map  $\eta_k : \mathbf{W}_{[n-1]}^k \rightarrow \Omega^{n-k-1}\overline{\mathbf{W}}^n$  satisfying (4.12), while by Lemma 4.4(b) we have a homotopy

$$(4.28) \quad H_{k-1} : \eta_k \circ d_{k-1}^0 \sim a_{k-1} : \mathbf{W}_{[n-1]}^{k-1} \rightarrow \Omega^{n-k-1}\overline{\mathbf{W}}^n$$

satisfying (4.17) for  $k-1$ .

We now define  $\widetilde{\mathbf{W}}_{[n]}^k$  by the pullback diagram (4.11), in which both vertical arrows are weak equivalences. To define the coface map  $\underline{d}_{k-1}^0 : \mathbf{W}_{[n-1]}^{k-1} \rightarrow \widetilde{\mathbf{W}}_{[n]}^k$  extending  $d_{k-1}^0 : \mathbf{W}_{[n-1]}^{k-1} \rightarrow \mathbf{W}_{[n-1]}^k$ , we set  $q^k \circ \underline{d}_{k-1}^0 : \mathbf{W}_{[n-1]}^{k-1} \rightarrow (\Omega^{n-k-1}\overline{\mathbf{W}}^n)^{\Delta[1]}$  equal to  $H_{k-1}$ .

To define the face map  $\underline{d}_k^1 : \widetilde{\mathbf{W}}_{[n]}^k \rightarrow \widetilde{\mathbf{W}}_{[n]}^{k+1}$  into the pullback (extending the given map  $d_k^1 \circ \psi_{[n]}^k : \mathbf{W}_{[n-1]}^k \rightarrow \mathbf{W}_{[n-1]}^{k+1}$ ), it suffices to specify the composite

$$q^{k+1} \circ \underline{d}_k^1 : \widetilde{\mathbf{W}}_{[n]}^k \rightarrow (\Omega^{n-k-2}\overline{\mathbf{W}}^n)^{\Delta[1]},$$

which we set equal to the composite:

$$(4.29) \quad \widetilde{\mathbf{W}}_{[n]}^k \xrightarrow{q^k} (\Omega^{n-k-1}\overline{\mathbf{W}}^n)^{\Delta[1]} \xrightarrow{\text{ev}_1} \Omega^{n-k-1}\overline{\mathbf{W}}^n \xrightarrow{i_k} (\Omega^{n-k-2}\overline{\mathbf{W}}^n)^{\Delta[1]}.$$

This indeed defines a map into the pullback (4.11) for  $k+1$ , since the composite

$$\Omega^{n-k-1}\overline{\mathbf{W}}^n \xrightarrow{i_k} (\Omega^{n-k-2}\overline{\mathbf{W}}^n)^{\Delta[1]} \xrightarrow{\text{ev}_0} \Omega^{n-k-2}\overline{\mathbf{W}}^n$$

is zero, which matches (4.12).

The remaining face maps  $\underline{d}_k^i : \underline{\mathbf{W}}_{[n]}^k \rightarrow \underline{\mathbf{W}}_{[n]}^{k+1}$  ( $2 \leq i \leq k+1$ ) are defined by extending the given  $d_k^i : \mathbf{W}_{[n-1]}^k \rightarrow \mathbf{W}_{[n-1]}^{k+1}$  by the zero map into  $(\Omega^{n-k-2}\overline{\mathbf{W}}^n)^{\Delta[1]}$  (which again matches (4.12)).

To verify the cosimplicial identity

$$(4.30) \quad \underline{d}_k^1 \circ \underline{d}_{k-1}^0 = \underline{d}_k^0 \circ \underline{d}_{k-1}^0 : \underline{\mathbf{W}}_{[n]}^{k-1} \rightarrow \underline{\mathbf{W}}_{[n]}^{k+1},$$

by (4.13) it suffices to check that  $\underline{d}_k^1 \underline{d}_{k-1}^0 = \underline{d}_k^0 \underline{d}_{k-1}^0$ , and for this in turn it suffices to check the post-composition with  $q^{k+1}$ , where:

$$\begin{aligned} q^{k+1} \circ \underline{d}_k^1 \circ \underline{d}_{k-1}^0 &= i_k \circ \text{ev}_1 \circ q^k \circ \underline{d}_{k-1}^0 = i_k \circ \text{ev}_1 \circ H_{k-1} \\ &= i_k \circ a_{k-1} = H_k \circ d_{k-1}^0 = q^{k+1} \circ \underline{d}_k^0 \circ d_{k-1}^0 = q^{k+1} \circ d_k^0 \circ \underline{d}_{k-1}^0, \end{aligned}$$

by (4.29), (4.28), (4.27), and (4.13).

The identities  $\underline{d}_{k+1}^j \underline{d}_k^i = \underline{d}_{k+1}^i \underline{d}_k^{j-1}$  hold trivially, with both sides vanishing, for all for  $k+2 \geq j > i \geq 0$  except for  $(j, i) \in \{(1, 0), (2, 0), (2, 1)\}$ . We check these three cases:

- (a) The case  $\underline{d}_{k+1}^1 \underline{d}_k^0 = \underline{d}_{k+1}^0 \underline{d}_k^0$  is (4.30), which was already verified in step  $k+1$ .
- (b) To show  $\underline{d}_{k+1}^2 \underline{d}_k^1 = \underline{d}_{k+1}^1 \underline{d}_k^1$ , it suffices to check the post-composition with  $q^{k+2}$ , where  $q^{k+2} \circ \underline{d}_{k+1}^2 = 0$  by definition, while

$$q^{k+2} \circ \underline{d}_{k+1}^1 \circ \underline{d}_k^1 = i_{k+1} \circ \text{ev}_1 \circ q^{k+1} \circ \underline{d}_k^1 = i_{k+1} \circ \text{ev}_1 \circ i_k \circ \text{ev}_1 \circ q^k = 0$$

since  $\text{ev}_1 \circ i_k : \Omega^{n-k-1}\overline{\mathbf{W}}^n \rightarrow \Omega^{n-k-2}\overline{\mathbf{W}}^n$  is the zero map.

- (c) To show  $\underline{d}_{k+1}^2 \underline{d}_k^0 = \underline{d}_{k+1}^0 \underline{d}_k^1$ , it suffices to check the post-composition with  $q^{k+2}$ , where again  $q^{k+2} \circ \underline{d}_{k+1}^2 = 0$  by definition, while

$$q^{k+2} \circ \underline{d}_{k+1}^0 \circ \underline{d}_k^1 = H_{k+1} \circ \psi_{[n]}^{k+1} \circ \underline{d}_k^1 = H_{k+1} \circ d_k^1 \circ \psi_{[n]}^k = 0$$

(in the notation of (4.11)), by (4.17).

### Step $k = 0$ of the descending induction:

If  $\mathfrak{X} = \mathfrak{M}_{\Theta} \mathbf{Y}$ , the last step of the induction is no different from the general  $k$ , with  $\mathbf{W}_{[n-1]}^{-1} := \mathbf{Y}$ . However, in the general case we no longer have an object  $\mathbf{W}_{[n-1]}^{k-1}$  in  $\mathcal{C}$  for  $k = 0$ , so we must modify our construction somewhat:

By induction the homotopy  $H_0$  is a 1-simplex in  $(\mathfrak{M}_{\Theta} \mathbf{W}_{[n-1]}^0 \{\Omega^{n-2}\overline{\mathbf{W}}^n\})_1$ , (for which (4.17) is vacuous). Its simplicial face maps are  $d_0 H_0 = \eta_1 \circ d_0^0$  and  $d_1 H_0 = a_0$ , respectively.

Applying  $\varepsilon_{[n-1]} : \mathfrak{M}_{\Theta} \mathbf{W}_{[n-1]}^0 \rightarrow \mathfrak{X}$  to  $H_0$  yields a 1-simplex  $\varepsilon_{[n-1]}(H_0) \in \mathfrak{X}\{\Omega^{n-2}\overline{\mathbf{W}}^n\}$  with

$$\begin{aligned} d_0 \varepsilon_{[n-1]}(H_0) &= \varepsilon_{[n-1]}(\eta_1 \circ d_0^0) = (\varepsilon_{[n-1]} \circ d_0^0)_{\#}(\eta_1) = (\varepsilon_{[n-1]} \circ d_0^1)_{\#}(\eta_1 \circ \overline{\varphi}_{[n-1]}^0) \\ &= \varepsilon_{[n-1]}(\eta_1 \circ d_0^1) = 0 \end{aligned}$$

by (4.10) and (4.12).

Similarly, since  $i_1 \circ a_0 = H_1 \circ d_0^0 : \mathbf{W}_{[n-1]}^0 \rightarrow (\Omega^{n-3}\overline{\mathbf{W}}^n)^{\Delta[1]}$  for  $i_1 : \Omega^{n-2}\overline{\mathbf{W}}^n \hookrightarrow (\Omega^{n-3}\overline{\mathbf{W}}^n)^{\Delta[1]}$ , we see that:

$$\begin{aligned} (i_1)_\# d_1 \varepsilon_{[n-1]}(H_0) &= d_1 \varepsilon_{[n-1]}(i_1 \circ H_0) = \varepsilon_{[n-1]}(i_1 \circ a_0) = \varepsilon_{[n-1]}(H_1 \circ d_0^0) \\ &= (\varepsilon_{[n-1]} \circ d_0^0)_\#(H_1) = (\varepsilon_{[n-1]} \circ d_0^1)_\#(H_1) = \varepsilon_{[n-1]}(H_1 \circ d_0^1) = 0 \end{aligned}$$

by (4.17).

Since  $i_1$  is a cofibration in  $\mathcal{C}$  by (2.10) (and the fact that any  $\overline{\mathbf{W}}^n \in \Theta$  is cofibrant), by §2.11(c)  $(i_1)_\#$  is monic, so  $d_1 \varepsilon_{[n-1]}(H_0) = 0$ , too.

Thus the 1-simplex  $\varepsilon_{[n-1]}(H_0) \in \mathfrak{X}\{\Omega^{n-2}\overline{\mathbf{W}}^n\}$  actually defines a 0-simplex  $a_{-1}$  in  $\mathfrak{X}\{\Omega^{n-1}\overline{\mathbf{W}}^n\}$ , representing some class  $[a_{-1}] \in \pi_0 \mathfrak{X}\{\Omega^{n-1}\overline{\mathbf{W}}^n\} = \Lambda\{\Omega^{n-1}\overline{\mathbf{W}}^n\}$ .

Because  $V_\bullet$  is a  $\Theta$ -algebra resolution of  $\Lambda$ ,  $\varepsilon : V_0 \rightarrow \Lambda$  is surjective, so there is a class  $[\eta_0] \in V_0\{\Omega^{n-1}\overline{\mathbf{W}}^n\}$  with  $\varepsilon([\eta_0]) = [a_{-1}]$ , which we can represent by a map  $\eta_0 : \mathbf{W}_{[n-1]}^0 \rightarrow \Omega^{n-1}\overline{\mathbf{W}}^n$ , together with a 1-simplex  $H_{-1} \in \mathfrak{X}\{\Omega^{n-1}\overline{\mathbf{W}}^n\}$  with  $d_0 H_{-1} = \varepsilon_{[n-1]}(\eta_0)$  and  $d_1 H_{-1} = a_{-1}$ .

We can thus use  $\eta_0$  to define  $\mathbf{W}_{[n]}^0$  by the pullback diagram (4.11), and use  $H_{-1}$  to extend  $\varepsilon_{[n-1]}$  to  $\varepsilon_{[n]} : \mathfrak{M}_\Theta \mathbf{W}_{[n]}^0 \rightarrow \mathfrak{X}$ .

### Completing the $n$ -th step of the outer induction:

Once the inner (descending) induction is complete, we define a full cosimplicial object  $\widehat{\mathbf{W}}_{[n]}^\bullet$  by ascending induction on the cosimplicial dimension  $k$ . This will be equipped with a map of restricted cosimplicial objects  $g : \widehat{\mathbf{W}}_{[n]}^\bullet \rightarrow \mathbf{W}_{[n]}^\bullet$ , such that the composite map  $f := \psi_{[n]} \circ g : \widehat{\mathbf{W}}_{[n]}^\bullet \rightarrow \mathbf{W}_{[n-1]}^\bullet$  is a map of cosimplicial objects which is a dimensionwise weak equivalence.

We start with  $\widehat{\mathbf{W}}_{[n]}^0 := \mathbf{W}_{[n]}^0$  and  $g^0 := \text{Id}$ , and define  $\widehat{\mathbf{W}}_{[n]}^k$  by the pullback diagram:

$$(4.31) \quad \begin{array}{ccc} \widehat{\mathbf{W}}_{[n]}^k & \xrightarrow[\boxed{\text{PB}}]{\zeta_{[n]}^k} & M^k \widehat{\mathbf{W}}_{[n]}^\bullet \\ \downarrow g^k & & \downarrow M^k f \\ \mathbf{W}_{[n]}^k & \xrightarrow{\psi_{[n]}^k} \mathbf{W}_{[n-1]}^k \xrightarrow{\zeta_{[n-1]}^k} & M^k \mathbf{W}_{[n-1]}^\bullet \end{array}$$

(using the notation of §1.2).

The codegeneracy maps of  $\widehat{\mathbf{W}}_{[n]}^\bullet$  are defined by (1.4), while the coface map  $\hat{d}_{k-1}^j : \widehat{\mathbf{W}}_{[n]}^{k-1} \rightarrow \widehat{\mathbf{W}}_{[n]}^k$  is defined by the maps  $\hat{d}_{k-1}^j : \mathbf{W}_{[n]}^{k-1} \rightarrow \mathbf{W}_{[n]}^k$  constructed in the descending induction and the induced map  $M^{k-1} \widehat{\mathbf{W}}_{[n]}^\bullet \rightarrow M^k \widehat{\mathbf{W}}_{[n]}^\bullet$  determined by the current induction, (1.4), and the cosimplicial identities.

Finally, we let  $h : \mathbf{W}_{[n]}^\bullet \xrightarrow{\sim} \widehat{\mathbf{W}}_{[n]}^\bullet$  be any Reedy cofibrant replacement (cf. §1.16). Note that  $\mathbf{W}_{[n]}^\bullet$  is still a weak CW object, with CW basis  $(\overline{\mathbf{W}}^k)_{k=0}^n$ , and the map  $\overline{\varphi}_{[n]}^k : \mathbf{W}_{[n]}^k \rightarrow \overline{\mathbf{W}}^k$  of (1.13) defined to be the composite:

$$\mathbf{W}_{[n]}^k \xrightarrow{h^k} \widehat{\mathbf{W}}_{[n]}^k \xrightarrow{g^k} \mathbf{W}_{[n]}^k \xrightarrow{\psi_{[n]}^k} \mathbf{W}_{[n-1]}^k \xrightarrow{\overline{\varphi}_{[n-1]}^k} \overline{\mathbf{W}}^k.$$



Since  $h$ ,  $g$ , and  $\psi_{[n]}$  are maps of (restricted) cosimplicial objects, by construction (and (4.17)) we see that (1.14) is indeed satisfied.

This completes the  $n$ -th step of the outer induction, and thus the proof of the Theorem.  $\square$

**4.32. Cosimplicial resolutions and higher operations.** The particular construction used to produce the realization  $\mathbf{W}^\bullet$  of the algebraic resolution  $V_\bullet \rightarrow \Lambda$  actually encodes, in an explicit form, the additional higher order information that is needed to distinguish between different objects  $\mathbf{Y}$  realizing the given  $\Theta$ -algebra  $\Lambda$ .

For example, if  $\mathcal{C} = \mathcal{T}_*$  and  $\mathcal{A} = (\mathbf{K}(R, n))_{n=0}^\infty$ , this additional data takes the form of the higher order cohomology operations (see, e.g., [Ad, BM, Ma, Wa]), as follows:

Note that the  $n$ -th object  $\overline{V}_n$  of the algebraic CW basis for  $V_\bullet$  is a coproduct of free monogenic  $\Theta_R$ -algebras of the form  $H^*(\mathbf{K}(R, n_i); R)$ , each of which is indexed by a map  $\phi_i : H^*(\mathbf{K}(R, n_i); R) \rightarrow V_{n-1}$  (the restriction of the attaching map  $\overline{\partial}_0^n : \overline{V}_n \rightarrow V_{n-1}$ ). Moreover,  $\phi_i$  factors through some finite coproduct  $\coprod_{j=1}^{k_i} H^*(\mathbf{K}(R, n_j); R)$  of free monogenic  $\Theta_R$ -algebras in  $V_{n-1}$ , each of which is in turn indexed by a map  $\psi_{i,j} : H^*(\mathbf{K}(R, n_j); R) \rightarrow V_{n-2}$ , and so on. Thus the original summand  $H^*(\mathbf{K}(R, n_i); R)$  of  $\overline{V}_n$  is ultimately indexed by a composable sequence of  $n+1$  maps between finitely generated free  $\Theta_R$ -algebras, except for the very last, which lands in  $H^*(\mathbf{Y}; R) = \pi_0 \mathfrak{X}$ .

Moreover, the composite  $(\coprod_{j=1}^{k_i} \psi_{i,j}) \circ \phi_i$  vanishes, since  $d_0 \circ \overline{\partial}_0^n$  is a CW object. This means that when we realize  $V_\bullet$  by the cosimplicial space  $\mathbf{W}^\bullet$ , the corresponding composite will be null-homotopic. This is the source of the map  $\eta_{n-2} : \overline{\mathbf{W}}_{[n-1]}^{n-2} \rightarrow \Omega \overline{\mathbf{W}}^n$  (in step  $k = n-2$  of the descending induction in the proof), which is in fact just the Toda bracket, or secondary cohomology operation, associated to this nullhomotopy (see [Ha, §4.1]). The 0-th face map into the higher loops  $\Omega^{n-k-1} \overline{\mathbf{W}}^n$  are analogously associated to higher order cohomology operations, corresponding to the initial segments of the above composable sequence of homotopy classes of maps indexing each summand  $H^*(\mathbf{K}(R, n_i); R)$ .

See [BJT] for a detailed discussion of higher homotopy operations and simplicial spaces in the dual setting.

## 5. REALIZING MAPPING ALGEBRAS

We are now in a position to address the fundamental question of recognizing mapping spaces of the form  $X = \text{map}_*(\mathbf{Y}, \mathbf{A})$ , where the space  $\mathbf{A}$  is given. Unfortunately, we are able to give a satisfactory answer only when  $\mathbf{A}$  is  $\mathbf{K}(R, i)$  for  $R = \mathbb{Q}$  or  $\mathbb{F}_p$ .

First, we note the following elementary fact:

**5.1. Lemma.** *If  $R$  is a field, and  $G \rightarrow B^\bullet$  is a coaugmented cosimplicial  $R$ -module such that the dual augmented simplicial  $R$ -module  $(B^\bullet)^\dagger \rightarrow G^\dagger$  is acyclic, then  $G \rightarrow B^\bullet$  is acyclic, too.*

Here  $W^\dagger := \text{Hom}_R(W, R)$  is the  $R$ -dual of an  $R$ -module  $W$ .

*Proof.* Write  $A_\bullet$  for  $(B^\bullet)^\dagger$ . By Definition 1.6, the  $n$ -th Moore chains object  $C_n A_\bullet$  of  $A_\bullet \rightarrow G^\dagger$  is obtained by applying  $\text{Hom}_R(-, R)$  to the  $n$ -th Moore cochains

object  $C^n B^\bullet$  of  $G \rightarrow B^\bullet$ , since the latter is defined by the colimit (1.8), and the former by the corresponding limit (1.7).

If we think of the cochain complex  $C^* B^\bullet$  as a negatively-indexed chain complex  $E_*$ , and thus of  $C_* A_\bullet$  as the cochain complex  $\text{Hom}_R(E_*, R)$ , we see that

$$0 = \pi_i A_\bullet = H_i(C_* A_\bullet) = H^{-i}(E_*, R) \xrightarrow{\cong} \text{Hom}_R(H_{-i}(E_*), R),$$

by the Universal Coefficient Theorem (cf. [We, Theorem 3.6.5]). Since the homology groups  $H_{-i}(E_*) \cong \pi^i B^\bullet$  are free  $R$ -modules, they must all vanish.  $\square$

**5.2. Definition.** For any ring  $R$ , a  $\Theta_R$ -algebra  $\Lambda$  is

- (a) *k-connected* if  $\Lambda\{\mathbf{K}(R, i)\} = 0$  for  $0 \leq i \leq k$ .
- (b) *Finite type* if  $\Lambda\{\mathbf{K}(R, n)\}$  is a finitely generated  $R$ -module for each  $n \geq 0$ .
- (c) *Finite* if it is finite type, and there is an  $N$  such that  $\Lambda\{\mathbf{K}(R, n)\}$  vanishes for  $n \geq N$ .
- (d) *Allowable* if there is a partially ordered set  $J$ , with the over category  $j/J$  finite for each  $j \in J$ , and a diagram of finite  $\Theta_R$ -algebras  $\underline{\Lambda} : J \rightarrow \Theta_R\text{-Alg}$  such that

$$(5.3) \quad \Lambda = \lim_{j \in J} \underline{\Lambda}_j$$

in  $\Theta_R\text{-Alg}$ .

We say that an  $\Theta_R$ -mapping algebra  $\mathfrak{X}$  is *simply-connected*, *finite type*, *finite*, or *allowable* if the  $\Theta_R$ -algebra  $\pi_0 \mathfrak{X}$  is such.

**5.4. Example.** Any finite-type  $\Theta_R$ -algebra  $\Lambda$  is allowable, with (5.3) given by the directed system of finite truncations of  $\Lambda$ .

**5.5. Remark.** If  $R$  is a field with  $|R| \leq \aleph_0$ , such as  $\mathbb{F}_q$  or  $\mathbb{Q}$ , and  $\dim_R V = \aleph_0$ , then  $V \cong \bigoplus_{i=1}^{\infty} R$ , so  $V^\dagger \cong \prod_{i=1}^{\infty} R$  and thus  $|V^\dagger| = |R|^{\aleph_0} > |R| \cdot \aleph_0$ . Therefore, no  $R$ -module  $W$  of dimension  $\aleph_0$  can be isomorphic to  $V^\dagger$  for any  $R$ -module  $V$ . Thus a  $\Theta_R$ -algebra  $\Lambda$  for which some  $\Lambda\{\mathbf{K}(R, n)\}$  is  $\aleph_0$ -dimensional is not allowable. Similar phenomena occur for other fields and other cardinals.

**5.6. Lemma.** When  $R$  is a field, and  $\Lambda$  is an allowable  $\Theta_R$ -algebra, there is a graded  $R$ -module  $M_*$  with  $\Lambda\{\mathbf{K}(R, i)\} \cong M_i^\dagger$  (as  $R$ -modules) for each  $i \geq 0$ .

*Proof.* Since (co)limits in functor categories are defined object-wise, this follows from the fact that for a finite dimensional  $R$ -module  $W$ ,  $W^\dagger$  has a natural isomorphism  $(W^\dagger)^\dagger \cong W$ , and thus

$$(5.7) \quad \begin{aligned} \Lambda\{\mathbf{K}(R, i)\} &= (\lim_{j \in J} \underline{\Lambda}_j)\{\mathbf{K}(R, i)\} \cong \lim_{j \in J} (U \underline{\Lambda}_j\{\mathbf{K}(R, i)\}^\dagger)^\dagger \\ &\cong (\text{colim}_{j \in J} U \underline{\Lambda}_j\{\mathbf{K}(R, i)\}^\dagger)^\dagger, \end{aligned}$$

where  $U : \Theta_R\text{-Alg} \rightarrow \text{gr } R\text{-Mod}$  is the forgetful functor to graded  $R$ -modules, which creates all limits in  $\Theta_R\text{-Alg}$  since it is a right adjoint.  $\square$

**5.8. Lemma.** When  $R$  is a field, any realizable  $\Theta_R$ -mapping algebra  $\mathfrak{X} = \mathfrak{M}_{\mathcal{A}} \mathbf{Y}$  is allowable, and the graded module  $M_*$  of Lemma 5.6 is  $H_*(\mathbf{Y}; R)$ .

*Proof.* When  $R$  is a field, we have:

$$(5.9) \quad H^i(\mathbf{Y}; R) \xrightarrow{\cong} \lim_{j \in J} H^i(\mathbf{Y}_j; R) \quad \text{for all } i \geq 0$$

by [HM], and  $\pi_0 \mathfrak{X}\{\mathbf{K}(R, i)\} = H^i(\mathbf{Y}, R) \cong \text{Hom}_R(H_i \mathbf{Y}, R)$  by the Universal Coefficient Theorem.  $\square$

In particular, any free  $\Theta_R$ -algebra is realizable, and thus allowable.

**5.10. Proposition.** *If  $R$  is a field of characteristic 0 or  $R = \mathbb{F}_p$ , let  $V_\bullet$  be any free simplicial  $\Theta_R$ -algebra resolution of an allowable  $\Theta_R$ -algebra  $\Lambda$ , with  $\varepsilon : V_0 \rightarrow \Lambda$  the augmentation, and let  $M_*$  and  $N_*$  be the graded  $R$ -modules of Lemma 5.6 for  $\Lambda$  and  $V_0$ , respectively. Then there is a graded  $R$ -linear map  $\psi_* : M_* \rightarrow N_*$  with  $\varepsilon\{\mathbf{K}(R, i)\} = \psi_i^\dagger$  for each  $i \geq 0$ .*

*Proof.* Since  $V_0 = H^*(\mathbf{W}^0; R)$  is a free  $\Theta_R$ -algebra, it is allowable, and by (5.3) we know that  $\varepsilon$  is determined by a compatible system of  $\Theta_R$ -algebra-maps  $\varepsilon_j : V_0 \rightarrow \Lambda_j$ . Moreover, if  $\mathbf{W}^0 = \prod_{\alpha \in A} \mathbf{K}(R, n_\alpha)$ , then  $V_0 \cong \coprod_{\alpha \in A} H^*(\mathbf{K}(R, n_\alpha); R)$  is a coproduct (over  $A$ ) of monogenic free  $\Theta_R$ -algebras. Thus  $\varepsilon_j : V_0 \rightarrow \Lambda_j$  is completely determined by choices of maps of  $\Theta_R$ -algebras

$$(5.11) \quad \varepsilon_{\alpha, j} : H^*(\mathbf{K}(R, n_\alpha); R) \rightarrow \Lambda_j ,$$

with compatibility requirements only with respect to the various  $j \in J$ .

Since both source and target in (5.11) are finite  $\Theta_R$ -algebras, such maps are completely determined by their duals, which are maps of  $\mathbb{F}_p$ -coalgebras over the mod  $p$  Steenrod algebra  $\varepsilon_{\alpha, j}^\dagger : \Lambda_j^\dagger \rightarrow H_*(\mathbf{K}(R, n_\alpha); R)$ .

When  $\text{char}(R) = 0$ , by the Milnor-Moore Theorem (cf. [MM, App.]) we have:

$$(5.12) \quad H_*(\mathbf{W}^0; R) = H_*\left(\prod_{\alpha \in A} \mathbf{K}(R, n_\alpha); R\right) \cong \bigotimes_{\alpha \in A} H_*(\mathbf{K}(R, n_\alpha); R) ,$$

where the right-hand side is the product in the category of cocommutative coalgebras (cf. [G, 1.1b]), since  $H_*(\mathbf{W}^0; R)$  is then the cofree cocommutative coalgebra  $V(G_*)$  on the graded  $R$ -module  $G_* := \pi_* \mathbf{W}^0$ , and the functor  $V$  preserves products since it is right adjoint to the forgetful functor.

When  $R = \mathbb{F}_p$ , (5.12) still holds by [Bo2, §4.4], where the right hand side is now the product in the category of  $\mathbb{F}_p$ -coalgebras over the Steenrod algebra. Therefore, all the maps  $\varepsilon_{\alpha, j}^\dagger$  define a unique map  $\varepsilon_j^\dagger : \Lambda_j^\dagger \rightarrow H_*(\mathbf{W}^0; R)$  as required.  $\square$

As noted in §0.1, the homotopy type of an  $R$ -GEM  $X \in \mathcal{S}_*$  alone does not allow us to determine whether it is of the form  $X \simeq \text{map}_*(\mathbf{Y}, \mathbf{K}(R, n))$  for some space  $\mathbf{Y} \in \mathcal{T}_*$ , and if so, to recover  $\mathbf{Y}$  uniquely – we need some extra structure. The natural additional structure to put on  $X$  (in the sense of §2.14) is that of an  $\Theta_R$ -mapping algebra. Thus for us, the mapping space recognition and realization problem is translated into an mapping algebra recognition and realization question for the  $\Theta_R$ -mapping algebra  $\mathfrak{X}$ :

**Question:** Is there space  $\mathbf{Y}$  such that  $\mathfrak{X} \simeq \mathfrak{M}_R \mathbf{Y}$ , and if so, is  $\mathbf{Y}$  uniquely determined?

The answer is given by the following:

**5.13. Theorem.** *When  $R$  is  $\mathbb{Q}$  or  $\mathbb{F}_p$ , any simply-connected allowable  $\Theta_R$ -mapping algebra  $\mathfrak{X}$  is weakly equivalent to  $\mathfrak{M}_R \mathbf{Y}$  (cf. §2.16) for a simply-connected  $R$ -complete space  $\mathbf{Y} \in \mathcal{T}_*$ , unique up to weak equivalence.*

*Proof.* Let  $V_\bullet \rightarrow \Lambda$  be any a free simplicial resolution of the allowable  $\Theta_R$ -algebra  $\Lambda$ . It is readily verified that if  $\Lambda$  is simply-connected, this resolution may be chosen so that each  $V_k$  is simply-connected, since  $\tilde{H}^i(K(R, n); R) = 0$  for  $0 \leq i < n$ .

By Theorem 4.6,  $V_\bullet$  may be realized by a cosimplicial space  $\mathbf{W}^\bullet$ , with each  $\mathbf{W}^n$  a simply-connected  $R$ -GEM, and the corresponding simplicial  $\Theta_R$ -mapping algebra  $\mathfrak{W}_\bullet := \mathfrak{M}_R \mathbf{W}^\bullet$  is augmented to  $\mathfrak{X}$ . We may assume that  $\mathbf{W}^\bullet$  is Reedy fibrant. Since  $V_\bullet \rightarrow \Lambda$  is acyclic, for each  $\mathbf{K}(R, i) \in \Theta_R$  we have:

$$(5.14) \quad \pi_n V_\bullet \{ \mathbf{K}(R, i) \} \cong \pi_n H^{i-n}(\mathbf{W}^\bullet; R) \cong \begin{cases} \Lambda \{ \mathbf{K}(R, i) \} & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by Lemma 5.1 the cosimplicial graded  $R$ -module  $H_*(\mathbf{W}^\bullet; R)$  satisfies:

$$(5.15) \quad \pi^n H_{i-n}(\mathbf{W}^\bullet; R) \cong \begin{cases} M_i & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

for  $M_*$  as in Lemma 5.6.

Therefore, because  $R = \mathbb{Q}$  or  $\mathbb{F}_p$ , the homology spectral sequence for the simplicial space  $\mathbf{W}^\bullet$  (cf. [A1, R, Bo2]), with:

$$(5.16) \quad E_{m,t}^2 = \pi^m H_t(\mathbf{W}^\bullet; R) \implies H_{t-m}(\text{Tot } \mathbf{W}^\bullet; R),$$

satisfies the hypotheses of [Bo2, Theorem 3.4], so it converges strongly, and  $\mathbf{Y} := \text{Tot } \mathbf{W}^\bullet$  is simply-connected. Moreover,  $H_*(\mathbf{Y}; R) \cong \Lambda^*$ , since (5.16) collapses at the  $E^2$ -term, so  $\mathbf{Y} \rightarrow \mathbf{W}^\bullet$  is a weak  $\mathcal{G}$ -resolution for  $\mathcal{G} = \text{Obj } \Theta_R$ , in the sense of [Bo1, Definition 6.1], which implies that  $\hat{L}_{\mathcal{G}} \mathbf{Y} \simeq \mathbf{Y}$  by [Bo1, Theorem 6.5] – that is,  $\mathbf{Y}$  is  $R$ -complete (cf. [BK1, I, §5] and [Bo1, §7.7]).

Note that the natural map  $\mathbf{W}^\bullet \rightarrow c(\mathbf{Y})^\bullet$  induces a weak equivalence of simplicial  $\Theta_R$ -mapping algebras  $\mathfrak{W}_\bullet \rightarrow c(\mathfrak{M}_R \mathbf{Y})_\bullet$  (cf. §2.21), where  $\mathfrak{W}_\bullet := \mathfrak{M}_R \mathbf{W}^\bullet$ . On the other hand, by Theorem 4.6 we also have a weak equivalence  $\mathfrak{W}_\bullet \rightarrow c(\mathfrak{X})_\bullet$ .

For any simplicial  $\Theta_R$ -mapping algebra  $\mathfrak{W}_\bullet$ , we can define its *realization*  $\mathfrak{Z} := \|\mathfrak{W}_\bullet\| : \Theta \rightarrow \mathcal{S}_*$  by letting  $\mathfrak{Z}\{\mathbf{B}\}$  denote the diagonal of the bisimplicial group  $\mathfrak{W}_\bullet\{\mathbf{B}\}$ , for any  $\mathbf{B} \in \Theta$ . Since  $P\mathfrak{Z}\{\mathbf{B}\} = \mathfrak{Z}\{P\mathbf{B}\} \rightarrow \mathfrak{Z}\{\mathbf{B}\}$  is a fibration by [A2, Theorem 6.2], and the diagonal preserves products and cofibrations, we see that  $\mathfrak{Z}$  satisfies the three conditions of Definition 2.11, so it is an  $\Theta_R$ -mapping algebra.

Moreover, by (5.15) the Quillen spectral sequence for the bisimplicial group  $\mathfrak{W}_\bullet\{\mathbf{B}\}$  collapses for any  $\mathbf{B} \in \Theta_R$ , and thus the natural maps of  $\Theta_R$ -mapping algebras

$$(5.17) \quad \mathfrak{M}_R \mathbf{Y} \longleftarrow \mathfrak{Z} = \|\mathfrak{W}_\bullet\| \longrightarrow \mathfrak{X}$$

induced by the simplicial augmentations  $\mathfrak{M}_R \mathbf{Y} \leftarrow \mathfrak{W}_\bullet \rightarrow \mathfrak{X}$  are both weak equivalences. Thus we see that  $\mathbf{Y}$  indeed realizes  $\mathfrak{X}$ , up to weak equivalence of  $\Theta_R$ -mapping algebras. The uniqueness up to weak equivalence (for  $R$ -complete  $\mathbf{Y}$ ) follows from [BK1, I, Lemma 5.5].  $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, 31905 HAIFA, ISRAEL  
*E-mail address:* blanc@math.haifa.ac.il, sen\_deba@math.haifa.ac.il